UTRECHT UNIVERSITY UNIVERSITY OF TRENTO

MASTER'S DEGREE IN MATHEMATICS

# Generalised Gluing and Exact Completion of Path Categories

*Supervisor*
Dr. Jaap van Oosten

*Candidate*
Matteo Spadetto

ACADEMIC YEAR 2018-2019

# Contents

**Introduction** **5**

**1 Exact Completion of a Path Category** **7**
1.1 Exact Completion of a Finitely Complete Category . . . . . . . . . . . . . . . 7
1.2 Exact Completion of a Path Category . . . . . . . . . . . . . . . . . . . . . . . 13
1.3 About the Exact Categories Obtained through these Procedures . . . . . . . 25

**2 Homotopy Natural Numbers in a Path Category** **35**

**3 Generalised Gluing for Path Categories** **41**
3.1 Grothendieck Fibrations and Fibred Path Categories . . . . . . . . . . . . . . 42
3.2 Generalised Gluing and its Notion of Homotopy . . . . . . . . . . . . . . . . . 47
3.3 Homotopy Natural Numbers in the Generalised Gluing . . . . . . . . . . . . . 62

**A Appendix** **73**
A.1 Basic Properties of Path Categories . . . . . . . . . . . . . . . . . . . . . . . . 73
A.2 Some Needed Lemmas and Remarks . . . . . . . . . . . . . . . . . . . . . . . . 77

**References** **81**

# Introduction

This thesis is mostly about the notions of *path category*, *exact completion* (of finitely complete categories and of path categories) and *generalised gluing* (of path categories). A path category is basically a category with a notion of *fibration* and a notion of *weak equivalence* satisfying a particular list of axioms. Through these two classes of arrows one can always define a relation of *homotopy* between arrows (on a given path category) such that the homotopy equivalences w.r.t. to this relation are precisely the weak equivalences. Moreover, this relation between parallel arrows of a given path category $\mathcal{C}$ happens to be an equivalence relation and agrees with the composition, hence there is a category $\mathrm{Ho}(\mathcal{C})$ (the so-called *homotopy category* of $\mathcal{C}$) whose objects are the ones of $\mathcal{C}$ and whose arrows are the classes of homotopic parallel arrows of $\mathcal{C}$. Essentially, the definition of path category is motivated by the fact that it provides models of an intensional type theory having propositional identity types. A typical feature usually enjoyed by categorical models of homotopy type theory is the existence of a weak factorisation system. However, in general a path category does not verify this property, as the factorisation of an arrow into a weak equivalence followed by a fibration is not unique, but it is just unique up to a stronger notion of homotopy, called *fibred homotopy.* Moreover, the particular exhibition, that is, the particular representative, of this homotopically unique factorization of a given arrow really matters. In other words, we really care about the explicit construction that we present of the factorisation of an arrow in a path category. The appendix contains many basic results about this and other concepts related to the theory of path categories.

The work is made of two parts. The first one corresponds to Chapter 1, which is devoted to the notion of exact completion. The first section is about the exact completion of a finitely complete category, while the second one regards the analogous version for path categories, showing how the former can be understood as an instance of the latter. Indeed every finitely complete category $\mathcal{C}$ admits a trivial structure of path category (where every arrow is a fibration and the class of isomorphism is the class of weak equivalences) such that the exact completion, according to the second section, of $\mathcal{C}$, together with this structure, is precisely its exact completion according to the first section.

We exhibit the construction of the free exact completion $\mathrm{Ex}(\mathcal{C})$ of a finitely complete category $\mathcal{C}$ as it is presented in [2]. Here the objects of $\mathrm{Ex}(\mathcal{C})$ are the pseudo equivalence relations in $\mathcal{C}$ on objects of $\mathcal{C}$ and the arrows between two pseudo equivalence relations are the equivalence classes (modulo the equivalence relation given by the pointwise pseudo equivalence relation) of arrows of $\mathcal{C}$ (between the supports of the given pseudo equivalence relations) which preserve the pseudo equivalence relation-structure. Instead, if $\mathcal{C}$ is a path category, its exact completion $\mathrm{Hex}(\mathcal{C})$ is obtained through the same procedure, with the only difference that the pseudo equivalence relations, constituting the objects of $\mathrm{Hex}(\mathcal{C})$, are also required to be fibrations of $\mathcal{C}$. We call them *homotopy equivalence relations.* Alternatively, given a path category $\mathcal{C}$, one can consider the category $\mathrm{Ex}'(\mathcal{C})$ whose objects are again the homotopy equivalence relations and whose arrows are simply the ones of $\mathcal{C}$ which agree with the homotopy equivalence relations defined on the source and the target, without the imposition that two pointwise equivalent arrows of $\mathcal{C}$ represent the same arrow of $\mathrm{Ex}'(\mathcal{C})$. It happens that there is a natural structure of path category over $\mathrm{Ex}'(\mathcal{C})$ such that the categories $\mathrm{Hex}(\mathcal{C})$ and $\mathrm{Ho}(\mathrm{Ex}'(\mathcal{C}))$ are equivalent.

In the third section of the first chapter we present a characterisation of the exact categories obtained by applying to a finitely complete category the procedure discussed in the first section. In particular we show that an exact category is the exact completion of a given finitely complete full subcategory if and only if the objects of latter are essentially the regular projective objects of the former and the former has enough projectives. In the remaining part of the section we study the analogous results for the exact categories obtained through the procedure of the second section and we conclude that, whenever $\mathcal{C}$ is a path category, then $\mathrm{Ho}(\mathcal{C})$ is a weakly finitely complete category whose exact completion, according to the first section, is precisely $\mathrm{Hex}(\mathcal{C})$.

The second part of the thesis consists of Chapter 2 and Chapter 3 and it regards the notion of *generalised gluing* for path categories. In Chapter 2 we give two definitions of homotopy natural numbers object in a path category (the second is a strengthening of the first) and we observe that it constitutes a natural numbers object up to homotopy, that is, an object which becomes a natural numbers object in the corresponding homotopy category. We also state and prove some criteria and characterization that we need in order to get the results of the last chapter.

In the first section of Chapter 3 we define the notion of *fibred path category*, which is essentially a Grothendieck fibration between two path categories agreeing with their structure. We observe that such a fibred path category enjoys some nice properties. For instance, it reflects the relation of (fibred) homotopy between parallel arrows (*homotopy lifting property*) and it preserves the (strong) homotopy natural numbers objects. In the second section we present the notion of *generalised gluing* for path categories. If we consider two functors between path categories (agreeing with their structure) sharing their codomain $\mathcal{D}$ and we assume that one of them is a fibred path category, then their generalised gluing is the full subcategory of their comma category spanned by those arrows that are fibrations in $\mathcal{D}$. We generalize some of the results contained in [12] by showing that the generalised gluing of such a pair of functors has a natural structure of path category and characterizing its notion of homotopy between parallel arrows. Finally, in the last section, we prove that, if the domains of this pair of functors have both the (strong) homotopy natural numbers objects, then their gluing has the (strong) homotopy natural numbers objects as well. This list of problems was proposed by Benno van den Berg in order to get a generalisation of the corresponding results stated in [12].

Certainly, the most important tool we make use of in the last chapter is the so-called *transport structure* of a fibration $Y \xrightarrow{f} X$ in a given path category, as it allows several crucial constructions. This notion models the concept of *transport* in homotopy type theory and informally, if $p$ is a path in $X$ from $f(y)$ to $x'$ for some points $y$ of $Y$ and $x'$ of $X$, then the transport structure produces a point $y'$ of $Y$ lying in the fiber of $x'$. We briefly discuss the notion of transport structure in the appendix.

*Brief acknowledgements.* From a personal point of view, many thanks to everyone with whom I have had a sincere contact during these authentic years of mathematics student. From the academic point of view, many thanks to Menno de Boer, since my thesis is directly based on his one; many thanks to Benno van den Berg for having suggested a concrete problem to work on; many thanks to Taichi Uemura for our discussion, and for his wise remarks, on his notion of *fibred path category*; many thanks to Jaap van Oosten for his supervision, for having introduced me to this subject and for having taught me a lot of beautiful things; many thanks to Prof. Stefano Baratella for having supervised me for three years and for having wisely suggested me every single choice I took.

# 1 Exact Completion of a Path Category

As we anticipated in the introduction, this chapter is about the concept of *exact completion*. In detail, we discuss it for finitely complete categories (first section) and path categories (second section), looking at the latter as a generalised version of the former.

## 1.1 Exact Completion of a Finitely Complete Category

In this section we summarize the proof contained in [2] that every category with finite limits has *the* free exact completion.

**Remark 1.1** (*Some recalls*)**.** We remind that a *regular category* is a finitely complete category whose kernel pairs have coequalizers and whose regular epimorphisms are stable under pullback. Equivalently, a regular category is a finitely complete category whose regular epimorphisms are stable under pullback and whose arrows have a regular epi-mono factorization.

If $\mathcal{C}$ is a regular category, we remind that an *equivalence relation* of $\mathcal{C}$ is a monomorphism $R \overset{\langle r_1, r_2\rangle}{\rightarrowtail} X \times X$ such that, for every object $A$ of $\mathcal{C}$, the subset of $\mathcal{C}(A, X) \times \mathcal{C}(A, X)$ whose elements are the couples $(f, g)$ such that $\langle f, g\rangle$ factors through $\langle r_1, r_2\rangle$ is an equivalence relation of the set $\mathcal{C}(A, X)$. Anyway, one can prove that a monomorphism $\langle r_1, r_2\rangle$ is an equivalence relation if and only if: $(r)$ the arrows $r_1$ and $r_2$ have a common section $X \xrightarrow{\rho} R$; $(s)$ there is an arrow $X \xrightarrow{\sigma} X$ such that $r_1\sigma = r_2$ and $r_2\sigma = r_1$; $(t)$ if the following square:

$$\begin{tikzcd} B \arrow[r, "q_2"] \arrow[d, "q_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"] & X \end{tikzcd}$$

is a pullback, then there is an arrow $B \xrightarrow{\tau} R$ such that $r_1q_1 = r_1\tau$ and $r_2q_2 = r_2\tau$. In this case, observe that the arrows $\rho$, $\sigma$ and $\tau$ are unique. We also remind that a *pseudo equivalence relation* of $\mathcal{C}$ is an arrow $R \xrightarrow{\langle r_1, r_2\rangle} X \times X$ satisfying the same property that, in case it is a monomorphism, it is required to satisfy in order to be an equivalence relation. That is, for every object $A$ of $\mathcal{C}$, the image of the map:

$$\mathcal{C}(A, R) \to \mathcal{C}(A, X) \times \mathcal{C}(A, X)$$
$$f \mapsto (r_1 \circ f, r_2 \circ f)$$

is an equivalence relation of the set $\mathcal{C}(A, X)$. Of course every equivalence relation is a pseudo equivalence relation. Moreover, as before an arrow $R \xrightarrow{\langle r_1, r_2\rangle} X \times X$ is a pseudo equivalence relation if and only if conditions $(r)$, $(s)$ and $(t)$ hold. In this case, observe that the arrows $\rho$, $\sigma$ and $\tau$ are not necessarily unique.

Let $\mathcal{C}$ be a regular category. Then, we remind that $\mathcal{C}$ is an *exact category* if and only if every equivalence relation is effective (i.e. a kernel pair) or, equivalently, if and only if the image of every pseudo equivalence relation is effective (i.e. a kernel pair). In fact, the equivalence relations of $\mathcal{C}$ are precisely the monomorphisms of the regular epi-mono factorization of its pseudo equivalence relations.

Let $\mathcal{C}$ be a finitely complete category. Let $\mathrm{Ex}(\mathcal{C})$ be the category defined as follows:

1. An object of $\mathrm{Ex}(\mathcal{C})$ is a couple $(X, R \xrightarrow{\langle r_1,r_2\rangle} X\times X)$, being $\langle r_1, r_2\rangle$ a pseudo equivalence relation of $\mathcal{C}$.

2. Whenever $(X, \langle r_1, r_2\rangle)$ and $(Y, \langle s_1, s_2\rangle)$ are couples as in 1., let us consider the arrows $X \xrightarrow{f} Y$ of $\mathcal{C}$ such that: whenever $A$ is an object of $\mathcal{C}$ and $x, y$ are arrows $A \to X$ such that $\langle x, y\rangle$ factors through $\langle r_1, r_2\rangle$, then $\langle fx, fy\rangle$ factors through $\langle s_1, s_2\rangle$. Observe that an arrow $X \xrightarrow{f} Y$ satisfies this property if and only if there is an arrow $R \xrightarrow{f'} S$ such that $(f \times f)\langle r_1, r_2\rangle = \langle s_1, s_2\rangle f'$. We say that two parallel arrows $f$ and $g$ satisfying this property are equivalent if and only if, for every object $A$ of $\mathcal{C}$ and every arrow $A \xrightarrow{x} X$, it is the case that $\langle fx, gx\rangle$ factors through $\langle s_1, s_2\rangle$. Equivalently, if and only if there is an arrow $X \to S$ such that $(X \to S \xrightarrow{s_1} Y) = f$ and $(X \to S \xrightarrow{s_2} Y) = g$. An arrow of $\mathrm{Ex}(\mathcal{C})$ with source $(X, \langle r_1, r_2\rangle)$ and target $(Y, \langle s_1, s_2\rangle)$ is an equivalence class (modulo this equivalence relation) of arrows $X \xrightarrow{f} Y$ of $\mathcal{C}$ satisfying the property of before.

3. The equivalence relation defined in 2. is clearly a congruence: if $f$ and $g$ are arrows $X \to Y$ of $\mathcal{C}$ representing arrows $(X, R) \to (Y, S)$ of $\mathrm{Ex}(\mathcal{C})$, and $\Sigma$ is an arrow $X \to S$ witnessing that $f$ and $g$ are pointwise equivalent, then, whenever $h$ and $k$ represent arrows to $(X, R)$ and from $(Y, S)$ respectively, it is the case that the arrows $\Sigma h$ and $k'\Sigma$ witness that $fh$ and $gh$ are pointwise equivalent and that $kf$ and $kg$ are pointwise equivalent, respectively (where $k'$ is an arrow from $S$ witnessing that $k$ agrees with the pseudo equivalence relations of the domain and the codomain). Then the composition is well-defined if we stipulate that the composition of two arrows of $\mathrm{Ex}(\mathcal{C})$ is represented by the composition in $\mathcal{C}$ of two representatives of them.

Let us consider the functor $\Gamma\colon \mathcal{C} \to \mathrm{Ex}(\mathcal{C})$ sending every object $X$ of $\mathcal{C}$ to the couple $(X, \langle 1_X, 1_X\rangle)$ and every arrow $X \xrightarrow{f} Y$ into the arrow $(X, \langle 1_X, 1_X\rangle) \to (Y, \langle 1_Y, 1_Y\rangle)$ represented by itself. This is clearly an embedding. We can state the following:

**Theorem 1.2.** *Let $\mathcal{C}$ be a finitely complete category. Then the functor $\Gamma\colon \mathcal{C} \to \mathrm{Ex}(\mathcal{C})$ is the free exact completion of $\mathcal{C}$. That is, the category $\mathrm{Ex}(\mathcal{C})$ is exact and the functor $\Gamma$ preserves finite limits and, whenever $\mathcal{D}$ is an exact category and $\Lambda\colon \mathcal{C} \to \mathrm{Ex}(\mathcal{C})$ is a finite limit preserving functor, there is essentially unique an* exact functor $\overline{\Lambda}\colon \mathrm{Ex}(\mathcal{C}) \to \mathcal{D}$ *(that is, $\overline{\Lambda}$ preserves finite limits and the regular epi-mono factorization of every arrow) such that the triangle:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Lambda"] \arrow[d, "\Gamma"'] & \mathcal{D} \\
\mathrm{Ex}(\mathcal{C}) \arrow[ur, "\overline{\Lambda}"'] &
\end{tikzcd}$$

*commutes.*

*Proof.* Firstly let us show that $\mathrm{Ex}(\mathcal{C})$ is finitely complete and that $\Gamma$ preserves finite limits. If 1 is the terminal object of $\mathcal{C}$ then, for every object $(X, R \xrightarrow{\langle r_1,r_2\rangle} X \times X)$ of $\mathrm{Ex}(\mathcal{C})$, the couple $(X \to 1, R \to 1)$ represents the unique arrow $(X, \langle r_1, r_2\rangle) \to (1, \langle 1_1, 1_1\rangle)$. Hence $(1, \langle 1_1, 1_1\rangle)$ is terminal object of $\mathrm{Ex}(\mathcal{C})$ and $\Gamma$ preserves the terminal object.

Let $[f]$ and $[g]$ be arrows:

$$(X, R \xrightarrow{\langle r_1,r_2\rangle} X\times X) \to (Z, T \xrightarrow{\langle t_1,t_2\rangle} Z\times Z) \text{ and } (Y, S \xrightarrow{\langle s_1,s_2\rangle} Y\times Y) \to (Z, T \xrightarrow{\langle t_1,t_2\rangle} Z\times Z)$$

of Ex($\mathcal{C}$) respectively. Let $T^* \xrightarrow{t^*} X \times Y$ be the pullback of the arrow $T \xrightarrow{\langle t_1, t_2 \rangle} Z \times Z$ along $f \times g$ and let $U \xrightarrow{u} T^* \times T^*$ be the pullback of the arrow $R \times S \xrightarrow{\langle r_1 \times s_1, r_2 \times s_2 \rangle} (X \times Y) \times (X \times Y)$ along $t^* \times t^*$. Then:

$$(T^*, U \xrightarrow{u} T^* \times T^*)$$

is an object of Ex($\mathcal{C}$) and the arrows $\pi_X t^*$ and $\pi_Y t^*$ represent arrows $(T^*, u) \to (X^*, \langle r_1, r_2 \rangle)$ and $(T_*, u) \to (Y^*, \langle s_1, s_2 \rangle)$ of Ex($\mathcal{C}$) respectively. Moreover the following square:

$$\begin{tikzcd} (T^*, u) \arrow[r, "{[\pi_Y t^*]}"] \arrow[d, "{[\pi_X t^*]}"'] & (Y, \langle s_1, s_2 \rangle) \arrow[d, "{[g]}"] \\ (X, \langle r_1, r_2 \rangle) \arrow[r, "{[f]}"'] & (Z, \langle t_1, t_2 \rangle) \end{tikzcd}$$

is a pullback in Ex($\mathcal{C}$). If $\langle r_1, r_2 \rangle = \langle 1_X, 1_X \rangle$, $\langle s_1, s_2 \rangle = \langle 1_Y, 1_Y \rangle$ and $\langle t_1, t_2 \rangle = \langle 1_Z, 1_Z \rangle$ then $(T^*, u) = (X \times_Z Y, \delta_{X \times_Z Y})$, $\pi_Y t^* = g^* f$ and $\pi_X t^* = f^* g$, where the following square:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "g^* f"] \arrow[d, "f^* g"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a pullback in $\mathcal{C}$. Hence $\Gamma$ preserves the pullbacks.

Secondly, in order to show that Ex($\mathcal{C}$) is regular, we observe that every arrow:

$$(X, R \xrightarrow{\langle r_1, r_2 \rangle} X \times X) \xrightarrow{[f]} (Y, S \xrightarrow{\langle s_1, s_2 \rangle} Y \times Y)$$

of Ex($\mathcal{C}$) factors as:

$$(X, R \xrightarrow{\langle r_1, r_2 \rangle} X \times X) \xrightarrow{[1_X]} (X, (f \times f)^* S \xrightarrow{(f \times f)^* \langle s_1, s_2 \rangle} X \times X) \xrightarrow{[f]} (Y, S \xrightarrow{\langle s_1, s_2 \rangle} Y \times Y)$$

and that $(X, R \xrightarrow{\langle r_1, r_2 \rangle} X \times X) \xrightarrow{[1_X]} (X, (f \times f)^* S \xrightarrow{\langle f^* s_1, f^* s_2 \rangle} X \times X)$ is a regular epimorphism of Ex($\mathcal{C}$) while $(X, (f \times f)^* S \xrightarrow{\langle f^* s_1, f^* s_2 \rangle} X \times X) \xrightarrow{[f]} (Y, S \xrightarrow{\langle s_1, s_2 \rangle} Y \times Y)$ is a monomorphism:

1. $[f]$ *is a monomorphism* $\langle f^* s_1, f^* s_2 \rangle \to \langle s_1, s_2 \rangle$. Let us assume that $[h]$ and $[g]$ are arrows $(T \xrightarrow{\langle t_1, t_2 \rangle} Z \times Z) \to \langle f^* s_1, f^* s_2 \rangle$ such that $[f][g] = [f][h]$. Then there is an arrow $Z \xrightarrow{\Sigma} S$ such that:

$$(f \times f)\langle g, h \rangle = \langle fg, fh \rangle = \langle s_1, s_1 \rangle \Sigma$$

and then, being the following square:

$$\begin{tikzcd} (f \times f)^* S \arrow[r] \arrow[d, "{\langle f^* s_1, f^* s_2 \rangle}"'] & S \arrow[d, "{\langle s_1, s_2 \rangle}"] \\ X \times X \arrow[r, "f \times f"] & Y \times Y \end{tikzcd}$$

a pullback, there is an arrow $Z \xrightarrow{\Sigma'} (f \times f)^* S$ such that $\langle g, h \rangle \Sigma' = \langle f^* s_1, f^* s_2 \rangle \Sigma'$, that is, $[g] = [h]$.

2. $[1_X]$ *is an epimorphism* $\langle r_1, r_2\rangle \to \langle f^*s_1, f^*s_2\rangle$. Of course, an arrow $R \to (f \times f)^*S$ witnessing that $1_X$ agrees with the equivalence relations exists by the universal property of the pullback:

$$\begin{tikzcd}
(f \times f)^*S \arrow[r] \arrow[d, "\langle f^*s_1, f^*s_2\rangle"'] & S \arrow[d, "\langle s_1, s_2\rangle"] \\
X \times X \arrow[r, "f \times f"] & Y \times Y
\end{tikzcd}$$

in $\mathcal{C}$ applied to the couple $(\langle r_1, r_2\rangle, f')$, where $f'$ is an arrow $R \to S$ witnessing that $f$ agrees with the equivalence relations.

We need to show that there is a parallel pair of arrows of $\mathrm{Ex}(\mathcal{C})$ whose coequalizer exists and is $[1_X]$. Let us consider the kernel pair of the arrow $\langle r_1, r_2\rangle \xrightarrow{[f]} \langle s_1, s_2\rangle$. Following the given construction of the pullbacks in $\mathrm{Ex}(\mathcal{C})$, we get that it consists of the following pair of arrows:

$$\begin{tikzcd}
U \arrow[r, "\pi_1 \bullet"] \arrow[d, "u"'] & R \arrow[d, "\langle r_1, r_2\rangle"] \\
f^*S \times f^*S \arrow[r, "f^*s_1 \times f^*s_1"] & X \times X
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\pi_2 \bullet"] \arrow[d, "u"'] & R \arrow[d, "\langle r_1, r_2\rangle"] \\
f^*S \times f^*S \arrow[r, "f^*s_2 \times f^*s_2"] & X \times X
\end{tikzcd}$$

being the couple $(u, \bullet)$ the pullback of the couple $(\langle f^*s_1, f^*s_2\rangle \times \langle f^*s_1, f^*s_2\rangle, \langle r_1, r_2\rangle)$ in $\mathcal{C}$. By definition of kernel pair, it is the case that $[f][f^*s_1] = [f][f^*s_2]$, that is, $([f][1_X])[f^*s_1] = ([f][1_X])[f^*s_2]$ (here $([f][1_X])$ denotes the factorization of $[f]$). As $[f]$ is monic, it is the case that $[1_X]$ coequalizes $[f^*s_1]$ and $[f^*s_2]$.

Let $[g]$ be an arrow $\langle r_1, r_2\rangle \to (T \xrightarrow{\langle t_1, t_2\rangle} Z \times Z)$ coequalizing the pair $([f^*s_1], [f^*s_2])$. We need to prove that it factors through $[1_X]$, that is, that $g$ also represents an arrow $\langle f^*s_1, f^*s_2\rangle \to \langle t_1, t_2\rangle$. But this is true: as $[g]$ coequalizes $[f^*s_1]$ and $[f^*s_2]$, there is an arrow $(f \times f)^*S \xrightarrow{x} T$ such that $\langle t_1, t_2\rangle x = \langle g(f^*s_1), g(f^*s_2)\rangle = (g \times g)\langle f^*s_1, f^*s_2\rangle$, that is, the following:

$$\begin{tikzcd}
(f \times f)^*S \arrow[r, "x"] \arrow[d, "\langle f^*s_1, f^*s_2\rangle"'] & T \arrow[d, "\langle t_1, t_2\rangle"] \\
X \times X \arrow[r, "g \times g"] & Z \times Z
\end{tikzcd}$$

commutes, and $[g]$ represents an arrow $\langle f^*s_1, f^*s_2\rangle \to \langle t_1, t_2\rangle$.

We explicitly constructed the regular epi-mono factorization of an arrow of $\mathrm{Ex}(\mathcal{C})$. As a consequence, the arrow $(X, R \xrightarrow{\langle r_1, r_2\rangle} X \times X) \xrightarrow{[f]} (Y, S \xrightarrow{\langle s_1, s_2\rangle} Y \times Y)$ is a regular epimorphism if and only if the arrow $(X, (f \times f)^*S \xrightarrow{(f\times f)^*\langle s_1, s_2\rangle} X \times X) \xrightarrow{[f]} (Y, S \xrightarrow{\langle s_1, s_2\rangle} Y \times Y)$ is an isomorphism. We can use this fact and the description that we gave of a pullback in $\mathrm{Ex}(\mathcal{C})$ in order to show that every regular epimorphism of $\mathcal{C}$ is stable under pullback. Indeed, up to postcomposing to an isomorphism, a regular epimorphism is nothing but an arrow represented by an identity arrow of $\mathcal{C}$. Let $(X, S) \to (X, R)$ be such an arrow and let $(Y,T) \xrightarrow{[f]} (X,R)$ be an arrow of $\mathrm{Ex}(\mathcal{C})$. Let us consider the following pullbacks:

$$\begin{tikzcd}
(f\times 1_X)^*R \arrow[r] \arrow[d, "a"'] & R \arrow[d, "\langle r_1,r_2\rangle"] \\
Y\times X \arrow[r, "f\times 1_X"] & X\times X
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r] \arrow[d, "u"'] & T\times S \arrow[d, "\langle t_1\times s_1, t_2\times s_2\rangle"] \\
(f\times 1_X)^*R\times (f\times 1_X)^*R \arrow[r, "a\times a"] & (Y\times X)\times (Y\times X)
\end{tikzcd}$$

then the arrow $((f\times 1_X)^*R, U) \xrightarrow{[\pi_1 a]} (Y,T)$ is the pullback of the arrow $(X,S)\to(X,R)$ along $[f]$. With respect to the first pullback, let us consider the arrow $Y \xrightarrow{\langle 1_Y, f\rangle} Y\times X$ and an arrow $Y\to R$ whose postcomposition by $\langle r_1,r_2\rangle$ is $\langle f,f\rangle$ (it exists as $\langle r_1,r_2\rangle$ is an equivalence relation). Then there is an arrow $Y \xrightarrow{b} (f\times 1_X)^*R$ such that $ab=\langle 1_Y,f\rangle$, hence $(\pi_1 a)b = 1_Y$. We need to verify that $\pi_1 a$ represents a regular epimorphism $((f\times 1_X)^*R, U)\to (Y,T)$. Let us observe that the diagram:

$$\begin{tikzcd}
T \arrow[d, "\langle t_1,t_2\rangle"'] \arrow[rr, bend left, "1_T"] & (\pi_1 a\times \pi_1 a)^*T \arrow[r] \arrow[d] & T \arrow[d, "\langle t_1,t_2\rangle"] \\
Y\times Y \arrow[r, "g\times g"] & (f\times 1_X)^*R\times (f\times 1_X)^*R \arrow[r, "\pi_1 a\times \pi_1 a"] & Y\times Y
\end{tikzcd}$$

commutes, hence there is an arrow $T\to(\pi_1 a\times\pi_1 a)^*T$ making it commute. Therefore $g$ represents a section $(Y,T)\to((f\times 1_X)^*R,U)$ of the image $((f\times 1_X)^*R,U)\to(Y,T)$ of $[\pi_1 a]$, that is, the image of $[\pi_1 a]$ is an isomorphism (see Lemma A.16) and hence $[\pi_1 a]$ is a regular epimorphism. We conclude that in $\mathrm{Ex}(\mathcal{C})$ regular epimorphisms are stable under pullback.

Moreover, one can use the characterization that we saw of the notion of pseudo equivalence relation in order to construct a coequalizer $c$ in $\mathrm{Ex}(\mathcal{C})$ of a given pseudo equivalence relation of $\mathrm{Ex}(\mathcal{C})$. Then the image of the given pseudo equivalence relation turns out to be the kernel pair of $c$. Hence $\mathrm{Ex}(\mathcal{C})$ is exact.

Finally, suppose that $\Lambda$ is a functor as in the statement. We just define $\overline{\Lambda}$ as the unique functor $\mathrm{Ex}(\mathcal{C})\to\mathcal{D}$ sending every object $(X, R \xrightarrow{\langle r_1,r_2\rangle} X\times X)$ of $\mathrm{Ex}(\mathcal{C})$ to the coequalizer in $\mathcal{D}$ of the kernel pair $(\Lambda r_1, \Lambda r_2)$ and verify that it is exact and essentially unique (the verification that this functor is well-defined is identical to the one contained in the proof of Proposition 1.17). For instance, let us verify that $\overline{\Lambda}$ preserves the regular epi-mono factorization, that is, that it preserves monomorphisms and regular epimorphisms:

1. $\overline{\Lambda}$ *preserves monomorphisms.* Up to precomposing by an isomorphism, a monomorphism of $\mathrm{Ex}(\mathcal{C})$ is of the form $\langle f^*s_1, f^*s_2\rangle \xrightarrow{[f]} \langle s_1,s_2\rangle$ for some arrow $X\xrightarrow{f} Y$ of $\mathcal{C}$. In particular, the following diagram:

$$\begin{tikzcd}
(f\times f)^*S \arrow[r] \arrow[d, "\langle f^*s_1, f^*s_2\rangle"'] & S \arrow[d, "\langle s_1,s_2\rangle"] \\
X\times X \arrow[r, "f\times f"] & Y\times Y
\end{tikzcd}$$

is a pullback and hence the following diagram:

$$\begin{tikzcd}
(\Lambda f\times\Lambda f)^*S \arrow[r] \arrow[d, "{\langle(\Lambda f)^*(\Lambda s_1),(\Lambda f)^*(\Lambda s_2)\rangle}"'] & \Lambda S \arrow[d, "{\langle\Lambda s_1,\Lambda s_2\rangle}"] \\
\Lambda X\times\Lambda X \arrow[r, "\Lambda f\times\Lambda f"] & \Lambda Y\times\Lambda Y
\end{tikzcd}$$

is a pullback as well, because $\Lambda$ preserves finite limits.

With respect to the commutative diagram:

$$\begin{tikzcd}
(\Lambda f\times\Lambda f)^*S \arrow[r] \arrow[d, shift right, "(\Lambda f)^*(\Lambda s_1)"'] \arrow[d, shift left, "(\Lambda f)^*(\Lambda s_2)"] & \Lambda S \arrow[d, shift right, "\Lambda s_1"'] \arrow[d, shift left, "\Lambda s_2"] \\
\Lambda X \arrow[r, "\Lambda f"'] \arrow[d, "q"'] & \Lambda Y \arrow[d, "q'"] \\
\overline{\Lambda}(f\times f)^*S \arrow[r, "\overline{\Lambda}[f]"] & \overline{\Lambda}S
\end{tikzcd}$$

we only need to prove that $\overline{\Lambda}f$ is a monomorphism. Here $q$ and $q'$ are coequalizers of the parallel pairs in the diagram, by definition of $\overline{\Lambda}$. These parallel pairs are still pseudo equivalence relations because $\Lambda$ preserves finite limits.

Now, let $(\Lambda f\times\Lambda f)^*S \xrightarrow{e} \mathrm{im} \xrightarrow{m} \Lambda X\times\Lambda X$ and $\Lambda S \xrightarrow{e'} \mathrm{im}' \xrightarrow{m'} \Lambda Y\times\Lambda Y$ be regular epi-mono factorizations of the pseudo equivalence relations $\langle(\Lambda f)^*(\Lambda s_1),(\Lambda f)^*(\Lambda s_2)\rangle$ and $\langle\Lambda s_1,\Lambda s_2\rangle$ respectively. Then $m=\langle m_1,m_2\rangle$ and $m'=\langle m'_1,m'_2\rangle$ are equivalence relations in $\mathcal{D}$, hence kernel pairs, since $\mathcal{D}$ is exact. Moreover, being $e$ and $e'$ epimorphisms, it is the case that $q$ and $q'$ are coequalizers of the couples $(m_1,m_2)$ and $(m'_1,m'_2)$ respectively. Hence $(m_1,m_2)$ and $(m'_1,m'_2)$ are the kernel pairs of $q$ and $q'$ respectively, by Lemma A.15. Finally, since in a regular category every pullback preserves the regular epi-mono factorization, it is the case that the following square:

$$\begin{tikzcd}
\mathrm{im} \arrow[r] \arrow[d, "{\langle m_1,m_2\rangle}"'] & \mathrm{im}' \arrow[d, "{\langle m'_1,m'_2\rangle}"] \\
\Lambda X\times\Lambda X \arrow[r, "\Lambda f\times\Lambda f"] & \Lambda Y\times\Lambda Y
\end{tikzcd}$$

is a pullback.

Let us consider the following diagram:

$$\begin{tikzcd}
\mathrm{im} \arrow[r] \arrow[d, shift right, "m_1"'] \arrow[d, shift left, "m_2"] & \mathrm{im}' \arrow[d, shift right, "m'_1"'] \arrow[d, shift left, "m'_2"] \\
\Lambda X \arrow[r, "\Lambda f"'] \arrow[d, "q"'] & \Lambda Y \arrow[d, "q'"] \\
\overline{\Lambda}(f\times f)^*S \arrow[r, "\overline{\Lambda}[f]"] & \overline{\Lambda}S
\end{tikzcd}$$

and let $\overline{\Lambda}(f\times f)^*S \xrightarrow{\varepsilon} \mathrm{im}'' \xrightarrow{\mu} \overline{\Lambda}S$ be a regular epi-mono factorization of $\overline{\Lambda}f$ in $\mathcal{D}$. We are done if we prove that $\varepsilon$ is an isomorphism. In order to prove this, it is enough to prove that $\varepsilon q$ is a coequalizer of the pair $(m_1, m_2)$. Clearly $\varepsilon q$ is a coequalizer, as it is the composition of two regular epimorphisms. Hence, by Lemma A.14, we are done if we prove that $(m_1, m_2)$ is the kernel pair of $\varepsilon q$. Clearly $\varepsilon q$ coequalizes $m_1$ and $m_2$. Let us assume that $\varepsilon q$ coequalizes a pair $(\alpha, \beta)$ of parallel arrows $A \to \Lambda X$. Then it is the case that $q'(\Lambda f)\alpha = q'(\Lambda f)\beta$ and, being $(m_1', m_2')$ the kernel pair of $q'$, there is unique an arrow $A \xrightarrow{a} \mathrm{im}'$ such that:

$$(\Lambda f \times \Lambda f)\langle \alpha, \beta\rangle = \langle (\Lambda f)\alpha, (\Lambda f)\beta\rangle = \langle m_1' a, m_2' a\rangle = \langle m_1', m_2'\rangle a.$$

Hence, by the universal property of the pullback, there is an arrow $A \xrightarrow{a'} \mathrm{im}$ such that $\langle m_1, m_2\rangle a' = \langle \alpha, \beta\rangle$. It is unique as $\langle m_1, m_2\rangle$ is a monomorphism. We conclude that $(m_1, m_2)$ is the kernel pair of $\varepsilon q$.

2. $\overline{\Lambda}$ *preserves regular epimorphisms.* Up to postcomposing by an isomorphism, a regular epimorphism of $\mathrm{Ex}(\mathcal{C})$ is an arrow $(X, \langle s_1, s_2\rangle) \to (X, \langle r_1, r_2\rangle)$ represented by the identity arrow $1_X$ of $\mathcal{C}$, and its kernel pair is the pair of arrows:

$$(R, U \xrightarrow{\langle u_1, u_2\rangle} R \times R) \to (X, S \xrightarrow{\langle s_1, s_2\rangle} X \times X)$$

represented by $r_1$ and $r_2$ respectively, where $u_1 = \langle r_1, r_2\rangle^* s_1$ and $u_2 = \langle r_1, r_2\rangle^* s_2$.

With respect to the following diagram:

$$\begin{tikzcd}
\Lambda U \arrow[r, shift left, "\Lambda r_1'"] \arrow[r, shift right, "\Lambda r_2'"'] \arrow[d, shift right, "\Lambda u_1"'] \arrow[d, shift left, "\Lambda u_2"] & \Lambda S \arrow[r] \arrow[d, shift right, "\Lambda s_1"'] \arrow[d, shift left, "\Lambda s_2"] & \Lambda R \arrow[d, shift right, "\Lambda r_1"'] \arrow[d, shift left, "\Lambda r_2"] \\
\Lambda R \arrow[r, shift left, "\Lambda r_1"] \arrow[r, shift right, "\Lambda r_2"'] \arrow[d, "q_1"] & \Lambda X \arrow[r, equal] \arrow[d, "q_2"] & \Lambda X \arrow[d, "q_3"] \\
\overline{\Lambda}(X, U) \arrow[r, shift left, "\overline{\Lambda}[r_1]"] \arrow[r, shift right, "\overline{\Lambda}[r_2]"'] & \overline{\Lambda}(X, S) \arrow[r, "\overline{\Lambda}[1_X]"] & \overline{\Lambda}(X, R)
\end{tikzcd}$$

we are done if we prove that $\overline{\Lambda}[1_X]$ is the coequalizer of the pair $(\overline{\Lambda}[r_1], \overline{\Lambda}[r_2])$. Clearly $\overline{\Lambda}[1_X]$ coequalizes the pair $(\overline{\Lambda}[r_1], \overline{\Lambda}[r_2])$, because $\overline{\Lambda}$ is a functor and $[1_X]$ coequalizes the pair $([r_1], [r_2])$. Let $a$ be an arrow $\overline{\Lambda}(X, S) \to A$ coequalizing the pair $(\overline{\Lambda}[r_1], \overline{\Lambda}[r_2])$. We are done if we prove that $a$ factors uniquely through $\overline{\Lambda}[1_X]$. We observe that $aq_2$ coequalizes both the pairs $(\Lambda r_1, \Lambda r_2)$ and $(\Lambda s_1, \Lambda s_2)$. Hence, being $\langle 1_A, 1_A\rangle$ an equivalence relation, there is an arrow $\Lambda S \to A$ witnessing that $aq_2$ represents an arrow $(\Lambda X, \langle \Lambda s_1, \Lambda s_2\rangle) \to (A, \langle 1_A, 1_A\rangle)$ coequalizing the pair $([\Lambda r_1], [\Lambda r_2])$. Therefore there is an arrow $(\Lambda X, \Lambda R) \xrightarrow{[h]} (A, \langle 1_A, 1_A\rangle)$ such that $[h][1_X] = [aq_2]$. In other words, there is an arrow $\Lambda X \xrightarrow{h'} A$ such that $\langle 1_A, 1_A\rangle h' = \langle h1_X, aq_2\rangle$, which implies that $h = aq_2$. Moreover, as $[h]$ is an arrow $(\Lambda X, \Lambda S) \to (A, \langle 1_A, 1_A\rangle)$, it is the case that $aq_2 = h$ coequalizes the couple $(\Lambda r_1, \Lambda r_2)$. Then there is unique an arrow $\overline{\Lambda}(X, R) \xrightarrow{b} A$ such that $aq_2 = bq_3 = b\overline{\Lambda}[1_X]q_2$, which implies that $a = b\overline{\Lambda}[1_X]$. Clearly, if there is another arrow $b'$ such that $a = b'\overline{\Lambda}[1_X]$, then $b'q_3 = bq_3$, hence $b' = b$. We are done. Q.E.D.

## 1.2 Exact Completion of a Path Category

In this section we present the notion of exact completion of a so-called path category, as defined in [9], trying to mimic the one discussed in Section 1.1 for finitely complete categories.

Moreover, we are going to see that this new notion actually generalises the previous one. We recall the following:

**Definition 1.3.** Let $\mathcal{C}$ be a category with a terminal object. Let us assume that there are two classes of arrow of $\mathcal{C}$ (the elements of the first one will be called *fibrations* and the elements of the second one will be called *weak equivalences*; moreover the elements of the intersection of these two classes will be called *acyclic fibrations*) such that the following properties are satisfied:

1. The composition of two fibrations is a fibration as well.
2. Every pullback of a fibration exists and is a fibration as well.
3. Every pullback of an acyclic fibration is an acyclic fibration as well.
4. For every choice of arrows $f$, $g$ and $h$, if the compositions $gf$ and $hg$ exist and are weak equivalences, then $f$, $g$, $h$ and $hgf$ are weak equivalences as well.
5. Every isomorphism is an acyclic fibration and every acyclic fibration has a section.
6. For every object $X$ of $\mathcal{C}$ there is an object $PX$, called *path object on* $X$, together with a weak equivalence $X \xrightarrow{r} PX$ and a fibration $PX \xrightarrow{\langle s,t\rangle} X \times X$ such that:
$$(X \xrightarrow{r} PX \xrightarrow{\langle s,t\rangle} X \times X) = (X \xrightarrow{\delta_X = \langle 1_X, 1_X\rangle} X \times X).$$
7. Every arrow of target a terminal object 1 is a fibration.

Then we say that $\mathcal{C}$ together with the given classes of fibrations and weak equivalences is a *path category*.

**Remark 1.4.** Let $\mathcal{C}$ be a path category and let $X$ and $Y$ be objects of $\mathcal{C}$. As their product $(X \times Y, X \times Y \xrightarrow{\pi_X} X, X \times Y \xrightarrow{\pi_Y} Y)$ is such that the square:

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \arrow[d] \\ Y \arrow[r] & 1 \end{tikzcd}$$

is a pullback and as this pullback exists by 2. and 7. of Definition 1.3, it is the case that the product of $X$ and $Y$ exists, hence $\mathcal{C}$ has finite products. Moreover, by 2. of Definition 1.3, it is the case that $\pi_X$ and $\pi_Y$ are fibrations. Therefore, whenever an arrow $X \xrightarrow{\langle f,g\rangle} Y \times Z$ is a fibration, by 1. it is the case that $f$ and $g$ are fibrations as well.

**Example 1.5.** Any category $\mathcal{C}$ with finite limits together with the class of its arrows as class of fibrations and the class of its isomorphisms as class of weak equivalences is a path category.

Points 1., 2., 3., 5. and 7. of Definition 1.3 are clear. Now, let us assume that $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are arrows of $\mathcal{C}$ such that $gf$ and $hg$ are isomorphisms. Let $a$ and $b$ be their inverses respectively. We observe that $(ag)f = a(gf) = 1_A$ and that $f(ag) = b(hg)f(ag) = bh((gf)a)g = bhg = b(hg) = 1_B$, hence $f$ is an isomorphism. Then $g = (gf)f^{-1}$ is also an isomorphism and hence $h = (hg)g^{-1}$ is an isomorphism as well. Finally $hgf$ is an isomorphism and point 4. holds. Moreover, if $X$ is an object of $\mathcal{C}$, then the object $X$ together with the arrows $1_X$ and $\langle 1_X, 1_X\rangle$ provides a path object on $X$, hence point 6. holds.

We remind that in a path category $\mathcal{C}$ the class of the weak equivalences and the class of the fibrations almost form a weak factorization system of $\mathcal{C}$ (see Proposition A.5 and Theorem A.13). Moreover we recall that, even if $\mathcal{C}$ is not necessarily finitely complete, it is closed under the notion of *homotopy pullback* (see Definition A.7) and that every arrow of $\mathcal{C}$ has a transport structure and a connection (see Definition A.8 and Theorem A.9). Consult Appendix A.1 for more details. Let us give the following:

**Definition 1.6.** Let $\mathcal{C}$ be a path category and let $X$ be an object of $\mathcal{C}$. We say that an arrow $R \xrightarrow{\langle r_1, r_2\rangle} X \times X$ is a *homotopy equivalence relation* on $X$ if and only if it is a fibration and a pseudo equivalence relation (see Section 1.1).

A weaker version of Theorem A.12 can be used to get a proof of the following:

**Proposition 1.7.** *Let $\mathcal{C}$ be a path category and let $X$ be an object of $\mathcal{C}$. Let $(PX, r, \langle s,t\rangle)$ be a triple satisfying 6. of Definition 1.3. Then the fibration $PX \xrightarrow{\langle s,t\rangle} X \times X$ is a homotopy equivalence relation on $X$. Moreover, whenever $R \xrightarrow{\langle r_1, r_2\rangle} Y \times Y$ is a homotopy equivalence relation on $Y$ and $f$ is an arrow $X \to Y$, there is an arrow $PX \xrightarrow{h} R$ such that $\langle r_1, r_2\rangle h = (f \times f)\langle s,t\rangle$. In particular, whenever $R \xrightarrow{\langle r_1, r_2\rangle} X \times X$ is a homotopy equivalence relation on $X$, there is an arrow $PX \xrightarrow{h} R$ such that $\langle r_1, r_2\rangle h = \langle s,t\rangle$.*

*Proof.* The reflexivity of $\langle s,t\rangle$ is clear, as there is a common section $r$ of $s$ and $t$ by definition of path object. Moreover, let us observe that the diagram:

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "r"'] & PX \arrow[d, "\langle s,t\rangle"] \\ PX \arrow[r, "\langle t,s\rangle"] & X \times X \end{tikzcd}$$

commutes. Then, by Theorem A.12 (actually a weaker version is enough) we get the symmetry of the pseudo relation $\langle s,t\rangle$.

For the transitivity, let us consider the diagram:

$$\begin{tikzcd} X \arrow[drr, bend left, "r"] \arrow[ddr, bend right, "r"'] & & \\ & PX \times_X PX \arrow[r, "s'"] \arrow[d, "t'"'] & PX \arrow[d, "t"] \\ & PX \arrow[r, "s"] & X \end{tikzcd}$$

which is commutative. Therefore, by the universal property of the pullback, there is unique an arrow $X \to PX \times_X PX$ making it commute, that is, the following diagram:

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d] & PX \arrow[d, "\langle s,t\rangle"] \\ PX \times_X PX \arrow[r, "\langle st', ts'\rangle"] & X \times X \end{tikzcd}$$

commutes and hence we get the transitivity of $\langle s,t\rangle$ again by applying Theorem A.12 (or a weaker version) to this square. Observe indeed that $X \to PX \times_X PX$ is a weak equivalence by 3. and 4. of Definition 1.3.

Finally, if $\rho$ witnesses the reflexivity of a given homotopy equivalence relation $\langle r_1, r_2\rangle$ and $f$ is an arrow $X \to Y$, then the diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"'] & Y \arrow[r, "\rho"] & R \arrow[d, "\langle r_1,r_2\rangle"] \\
PX \arrow[r, "\langle s,t\rangle"] & X \arrow[r, "f\times f"] & Y \times Y
\end{tikzcd}$$

commutes and again we get the arrow $h$ we were looking for by Theorem A.12. Q.E.D.

If $R \xrightarrow{\langle r_1,r_2\rangle} X \times X$ is a homotopy equivalence relation of a path category $\mathcal{C}$ and $Y \xrightarrow{f} X$ is an arrow of $\mathcal{C}$, then we denote as $f^*R \xrightarrow{f^*\langle r_1,r_2\rangle} Y \times Y$ the pullback of the fibration $\langle r_1, r_2\rangle$ along $f \times f$.

Let $\mathcal{C}$ be a path category. Let $\mathrm{Hex}(\mathcal{C})$ be the category defined as follows:

1. An object of $\mathrm{Hex}(\mathcal{C})$ is a couple $(X, R \xrightarrow{\langle r_1,r_2\rangle} X \times X)$, being $\langle r_1, r_2\rangle$ a homotopy equivalence relation of $\mathcal{C}$.

2. Whenever $(X, \langle r_1, r_2\rangle)$ and $(Y, \langle s_1, s_2\rangle)$ are couples as in 1., let us consider the arrows $X \xrightarrow{f} Y$ of $\mathcal{C}$ such that there is an arrow $R \xrightarrow{f'} S$ such that $(f \times f)\langle r_1, r_2\rangle = \langle s_1, s_2\rangle f'$. We say that two parallel arrows $f$ and $g$ satisfying this property are equivalent if and only if there is an arrow $X \to S$ such that $(X \to S \xrightarrow{s_1} Y) = f$ and $(X \to S \xrightarrow{s_2} Y) = g$ i.e. $(X \to S \xrightarrow{\langle s_1,s_2\rangle} Y \times Y) = \langle f, g\rangle$. An arrow of $\mathrm{Hex}(\mathcal{C})$ with source $(X, \langle r_1, r_2\rangle)$ and target $(Y, \langle s_1, s_2\rangle)$ is an equivalence class (modulo this equivalence relation) of arrows $X \xrightarrow{f} Y$ of $\mathcal{C}$ satisfying the property of before.

3. The equivalence relation defined in 2. is a congruence. Then the composition is well-defined if we stipulate that the composition of two arrows of $\mathrm{Hex}(\mathcal{C})$ is represented by the composition in $\mathcal{C}$ of two representatives of them.

We are going to prove that the cateogory $\mathrm{Hex}(\mathcal{C})$ is left exact. As its definition is almost the definition of $\mathrm{Ex}(\mathcal{D})$ for a finitely complete category $\mathcal{D}$, we expect this proof to be almost the one of Theorem 1.2.

**Proposition 1.8.** *Let $\mathcal{C}$ be a path category. Then* $\mathrm{Hex}(\mathcal{C})$ *is regular.*

*Proof.* By Proposition 1.7, the object $(X, PX \xrightarrow{\langle s,t\rangle} X \times X)$ is a terminal object of $\mathrm{Hex}(\mathcal{C})$ whenever $X$ is a terminal object of $\mathcal{C}$ and $\mathcal{C}$ has a terminal object. Let $[f]$ be an arrow $(Y, S \xrightarrow{\langle s_1,s_2\rangle} Y \times Y) \to (X, R \xrightarrow{\langle r_1,r_2\rangle} X \times X)$ of $\mathrm{Hex}(\mathcal{C})$ and let $g$ be an arrow $(Z, T \xrightarrow{\langle t_1,t_2\rangle} Z \times Z) \to (X, R \xrightarrow{\langle r_1,r_2\rangle} X \times X)$ of $\mathrm{Hex}(\mathcal{C})$. Let $R^* \xrightarrow{r^*} Y \times Z$ be the pullback in $\mathcal{C}$ of the arrow $R \xrightarrow{\langle r_1,r_2\rangle} X \times X$ (it exists because $\langle r_1, r_2\rangle$ is a fibration) along $f \times g$. Let $(\pi_1 \times \pi_1)^*S \xrightarrow{\langle \pi_1^* s_1,\pi_1^* s_2\rangle} (Y \times Z) \times (Y \times Z)$ be the pullback of $\langle s_1, s_2\rangle$ along the arrow $\pi_1 \times \pi_1 \colon (Y \times Z) \times (Y \times Z) \to Y \times Y$ and let $(\pi_2 \times \pi_2)^*T \xrightarrow{\langle \pi_2^* t_1,\pi_2^* t_2\rangle} (Y \times Z) \times (Y \times Z)$ be the pullback of $\langle t_1, t_2\rangle$ along the arrow $\pi_2 \times \pi_2 \colon (Y \times Z) \times (Y \times Z) \to Z \times Z$. Let us consider their intersection, i.e. the unique arrow arrow $(\pi_1\times\pi_1)^*S\cap(\pi_2\times\pi_2)^*T \to (Y\times Z)\times(Y\times Z)$, which is a homotopy equivalence relation as well, given by the following pullback:

$$\begin{tikzcd}
(\pi_1\times\pi_1)^*S\cap(\pi_2\times\pi_2)^*T \arrow[r] \arrow[d] & (\pi_2\times\pi_2)^*T \arrow[d, "\langle \pi_2^*t_1,\pi_2^*t_2\rangle"] \\
(\pi_1\times\pi_1)^*S \arrow[r, "\langle \pi_1^*s_1,\pi_1^*s_2\rangle"] & (Y\times Z)\times(Y\times Z)
\end{tikzcd}$$

and $U \xrightarrow{u} R^*\times R^*$ be the pullback of the arrow $(\pi_1\times\pi_1)^*S\cap(\pi_2\times\pi_2)^*T \to (Y\times Z)\times(Y\times Z)$ along $r^*\times r^*$. Then $(R^*, U \xrightarrow{u} R^*\times R^*)$ is an object of $\mathrm{Ex}(\mathcal{C})$ and the arrows $\pi_Y r^*$ and $\pi_Z r^*$ represent arrows $(R^*,u)\to(Y^*,\langle s_1,s_2\rangle)$ and $(R_*,u)\to(Z^*,\langle t_1,t_2\rangle)$ of $\mathrm{Hex}(\mathcal{C})$ respectively. Moreover the following square:

$$\begin{tikzcd}
(R^*,u) \arrow[r, "\pi_Z r^*"] \arrow[d, "\pi_Y r^*"'] & (Z,\langle t_1,t_2\rangle) \arrow[d, "g"] \\
(Y,\langle s_1,s_2\rangle) \arrow[r, "f"'] & (X,\langle r_1,r_2\rangle)
\end{tikzcd}$$

is a pullback in $\mathrm{Hex}(\mathcal{C})$. Hence $\mathrm{Hex}(\mathcal{C})$ is finitely complete.

Observe that for a given arrow $(X, R \xrightarrow{\langle r_1,r_2\rangle} X\times X) \xrightarrow{[f]} (Y, S \xrightarrow{\langle s_1,s_2\rangle} Y\times Y)$ of $\mathrm{Hex}(\mathcal{C})$ it is the case that the following diagram:

$$\begin{tikzcd}
(X,\langle r_1,r_2\rangle) \arrow[rr, "{[f]}"] \arrow[dr, "{[1_X]}"'] & & (Y,\langle s_1,s_2\rangle) \\
& (X,(f\times f)^*\langle s_1,s_2\rangle) \arrow[ur, "{[f]}"'] &
\end{tikzcd}$$

commutes in $\mathrm{Hex}(\mathcal{C})$, where $(X,\langle r_1,r_2\rangle) \xrightarrow{[1_X]} (X, f^*\langle s_1,s_2\rangle)$ is a regular epimorphism and $(X, f^*\langle s_1,s_2\rangle) \xrightarrow{[f]} (Y,\langle s_1,s_2\rangle)$ is a monomorphism:

1. $[f]$ *is a monomorphism* $\langle f^*s_1, f^*s_2\rangle \to \langle s_1,s_2\rangle$. Let us assume that $[h]$ and $[g]$ are arrows $(T \xrightarrow{\langle t_1,t_2\rangle} Z\times Z) \to \langle f^*s_1,f^*s_2\rangle$ such that $[f][g]=[f][h]$. Then there is an arrow $Z \xrightarrow{\Sigma} S$ such that:

$$(f\times f)\langle g,h\rangle = \langle fg, fh\rangle = \langle s_1,s_1\rangle\Sigma$$

and then, being the following square:

$$\begin{tikzcd}
(f\times f)^*S \arrow[r] \arrow[d, "\langle f^*s_1,f^*s_2\rangle"'] & S \arrow[d, "\langle s_1,s_2\rangle"] \\
X\times X \arrow[r, "f\times f"] & Y\times Y
\end{tikzcd}$$

a pullback, there is an arrow $Z \xrightarrow{\Sigma'} (f\times f)^*S$ such that $\langle g,h\rangle\Sigma' = \langle f^*s_1, f^*s_2\rangle\Sigma'$, that is, $[g]=[h]$.

2. $[1_X]$ *is an epimorphism* $\langle r_1, r_2\rangle \to \langle f^*s_1, f^*s_2\rangle$. Of course, an arrow $R \to (f\times f)^*S$ witnessing that $1_X$ represents an arrow of $\mathrm{Hex}(\mathcal{C})$ exists by the universal property of the pullback:

$$\begin{tikzcd} (f\times f)^*S \arrow[r] \arrow[d, "\langle f^*s_1, f^*s_2\rangle"'] & S \arrow[d, "\langle s_1, s_2\rangle"] \\ X\times X \arrow[r, "f\times f"] & Y\times Y \end{tikzcd}$$

in $\mathcal{C}$ applied to the couple $(\langle r_1, r_2\rangle, f')$, where $f'$ is an arrow $R\to S$ witnessing that $f$ represents an arrow of $\mathrm{Hex}(\mathcal{C})$.

We need to show that there is a parallel pair of arrows of $\mathrm{Hex}(\mathcal{C})$ whose coequalizer exists and is $[1_X]$. Let us consider the kernel pair of the arrow $\langle r_1, r_2\rangle \xrightarrow{[f]} \langle s_1, s_2\rangle$, which consists of the following pair of arrows:

$$\begin{tikzcd} U \arrow[r, "a\bullet"] \arrow[d, "u"'] & R \arrow[d, "\langle r_0, r_1\rangle"] \\ f^*S\times f^*S \arrow[r, "f^*s_1\times f^*s_1"] & X\times X \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "b\bullet"] \arrow[d, "u"'] & R \arrow[d, "\langle r_0, r_1\rangle"] \\ f^*S\times f^*S \arrow[r, "f^*s_2\times f^*s_2"] & X\times X \end{tikzcd}$$

being the couple $(u, \bullet)$ the pullback of the couple:

$$(\langle f^*s_1, f^*s_2\rangle \times \langle f^*s_1, f^*s_2\rangle, (\pi_1\times\pi_1)^*R \cap (\pi_2\times\pi_2)^*R \to (X\times X)\times(X\times X))$$

in $\mathcal{C}$ and being $a$ and $b$ the arrows $(\pi_1\times\pi_1)^*R\cap(\pi_2\times\pi_2)^*R \to (\pi_1\times\pi_1)^*R \to R$ and $(\pi_1\times\pi_1)^*R\cap(\pi_2\times\pi_2)^*R \to (\pi_2\times\pi_2)^*R \to R$ respectively. By definition of kernel pair, it is the case that $[f][f^*s_1] = [f][f^*s_2]$, that is, $([f][1_X])[f^*s_1] = ([f][1_X])[f^*s_2]$ (here $([f][1_X])$ denotes the factorization of $[f]$). As $[f]$ is monic, it is the case that $[1_X]$ coequalizes $[f^*s_1]$ and $[f^*s_2]$.

Let $[g]$ be an arrow $\langle r_1, r_2\rangle \to (T \xrightarrow{\langle t_1, t_2\rangle} Z\times Z)$ coequalizing the pair $([f^*s_1], [f^*s_2])$. The arrow $g$ of $\mathcal{C}$ also represents an arrow $\langle f^*s_1, f^*s_2\rangle \to \langle t_1, t_2\rangle$, because $[g]$ coequalizes $[f^*s_1]$ and $[f^*s_2]$ and hence there is an arrow $(f\times f)^*S \xrightarrow{x} T$ such that $\langle t_1, t_2\rangle x = \langle g(f^*s_1), g(f^*s_2)\rangle = (g\times g)\langle f^*s_1, f^*s_2\rangle$, that is, the following:

$$\begin{tikzcd} (f\times f)^*S \arrow[r, "x"] \arrow[d, "\langle f^*s_1, f^*s_2\rangle"'] & T \arrow[d, "\langle t_1, t_2\rangle"] \\ X\times X \arrow[r, "g\times g"] & Z\times Z \end{tikzcd}$$

commutes. We conclude that $\langle r_1, r_2\rangle \xrightarrow{[g]} \langle t_1, t_2\rangle$ factors through $[1_X]$.

We explicitly constructed the regular epi-mono factorization of the arrow $(X, \langle r_1, r_2\rangle) \xrightarrow{[f]} (Y, \langle s_1, s_2\rangle)$, hence it is the case that $[f]$ is a monomorphism if and only if the regular epimorphism in its factorization is an isomorphism, that is, if and only if there is an arrow $(f\times f)^*S \xrightarrow{h} R$ of $\mathcal{C}$ such that $((f\times f)^*S \xrightarrow{h} R \xrightarrow{\langle r_1, r_2\rangle} X\times X) = (f\times f)^*\langle s_1, s_2\rangle$ and it is a regular epimorphism if and only if its image is an isomorphism, that is, if and only if there are arrows $Y \xrightarrow{g} X$ and $Y \xrightarrow{h} S$ of $\mathcal{C}$ such that $(Y \xrightarrow{h} S \xrightarrow{\langle s_1, s_2\rangle} Y\times Y) = \langle 1_Y, fg\rangle$ (here we used Lemma A.16). We can use this last characterization in order to get that a regular epimorphism of $\mathrm{Hex}(\mathcal{C})$ is stable under pullback. Up to postcomposing by an isomorphism of $\mathrm{Hex}(\mathcal{C})$, an epimorphism of $\mathrm{Hex}(\mathcal{C})$ is represented by an identity. Let $(X,R) \to (X,S)$ be such an arrow and let $(Y,T) \xrightarrow{[f]} (X,S)$ be an arrow of $\mathrm{Hex}(\mathcal{C})$. Then the pullback of $(X,R) \xrightarrow{[1_X]} (X,S)$ along $[f]$ is the arrow $(U \xrightarrow{u} (f\times 1_X)^*S \times (f\times 1_X)^*S) \to (T \to Y\times Y)$ of $\mathrm{Hex}(\mathcal{C})$ represented by the arrow:

$$(f\times 1_X)^*S \xrightarrow{s} Y\times X \xrightarrow{\pi_1} Y$$

of $\mathcal{C}$, where $s$ is given by the following pullback:

$$\begin{tikzcd}
(f\times 1_X)^*S \arrow[r] \arrow[d, "s"'] & S \arrow[d, "\langle s_1,s_2\rangle"] \\
Y\times X \arrow[r, "f\times 1_X"] & X\times X.
\end{tikzcd}$$

Let us consider the arrow $Y \xrightarrow{\langle 1_Y,f\rangle} Y\times X$ of $\mathcal{C}$ and let $r$ be an arrow $Y\to S$ such that $(f\times 1_X)\langle 1_Y,f\rangle = \langle f,f\rangle = \langle s_1,s_2\rangle r$, which exists since . By the universal property of the pullback, there is an arrow $Y \xrightarrow{g} (f\times 1_X)^*S$ such that $sg = \langle 1_Y,f\rangle$ and then $(\pi_1 s)g = 1_Y$. Being $T$ a homotopy equivalence relation, there is an arrow $h\colon Y\to T$ such that $\langle 1_Y,(\pi_1 s)g\rangle = \langle t_1,t_2\rangle h$. By the previous characterization of regular epimorphisms, we conclude that $[\pi_1 s]$ is a regular epimorphism. Hence regular epimorphisms are stable under pullback and $\mathrm{Hex}(\mathcal{C})$ is regular. Q.E.D.

**Remark 1.9.** Let $\mathcal{C}$ be a finitely complete category and let $\mathcal{D}$ be a path category. We observe that the constructions of the pullback of a couple of arrows in $\mathrm{Ex}(\mathcal{C})$ and $\mathrm{Hex}(\mathcal{D})$ (see Theorem 1.2 and Proposition 1.8 respectively) are formally different. If $(X, R \xrightarrow{\langle r_1,r_2\rangle} X\times X) \to (Z, T \xrightarrow{\langle t_1,t_2\rangle} Z\times Z)$ and $(Y, S \xrightarrow{\langle s_1,s_2\rangle} Y\times Y) \to (Z, T \xrightarrow{\langle t_1,t_2\rangle} Z\times Z)$ are arrows of $\mathrm{Ex}(\mathcal{C})$ and $W \xrightarrow{w} X\times Y$ is the pullback of $T\to Z\times Z$ along the arrow $X\times Y\to Z\times Z$, then we consider the pullback along $w\times w$ of the pseudo equivalence relation $R\times S \xrightarrow{\langle r_1\times s_1, s_2\times r_2\rangle} (X\times Y)\times(X\times Y)$, in order to get a pseudo equivalence relation over $W$. Instead, if $(X,R)\to (Z,T)$ and $(Y,S)\to(Z,T)$ are arrows of $\mathrm{Hex}(\mathcal{D})$, then we consider the pullback along $w\times w$ of the homotopy equivalence realtion $(\pi_1\times\pi_1)^*R \cap (\pi_2\times\pi_2)^*S \to (X\times Y)\times(X\times Y)$, where $\pi_1$ and $\pi_2$ are the projections $X\times Y\to X$ and $X\times Y\to Y$ respectively.

Actually, the pseudo relations $R\times S \xrightarrow{\langle r_1\times s_1, s_2\times r_2\rangle} (X\times Y)\times(X\times Y)$ and $(\pi_1\times\pi_1)^*R\cap(\pi_2\times\pi_2)^*S \to (X\times Y)\times(X\times Y)$ informally (set-theoretically) represent "the same equivalence relation". Indeed, let us assume that the regular images $\mathrm{im}(R)$ and $\mathrm{im}(S)$ (which are equivalence relations) of $R$ and $S$ exist in both $\mathcal{C}$ and $\mathcal{D}$, so that there exist formulas $\varphi(x_1 : X, x_2 : X)$ and $\psi(y_1 : Y, y_2 : Y)$ (here we use the same symbols for $\mathcal{C}$ and $\mathcal{D}$) in both the internal languages of $\mathcal{C}$ and $\mathcal{D}$, such that the subobject $\mathrm{im}(R)$ of $X\times X$ is the subobject $\{(x_1,x_2) : X\times X \mid \varphi(x_1,x_2)\}$ and the subobject $\mathrm{im}(S)$ of $Y\times Y$ is the subobject $\{(y_1,y_2) : Y\times Y \mid \psi(y_1,y_2)\}$. Then we could verify that the regular images of $R\times S \xrightarrow{\langle r_1\times s_1, s_2\times r_2\rangle} (X\times Y)\times(X\times Y)$ and $(\pi_1\times\pi_1)^*R\cap(\pi_2\times\pi_2)^*S \to (X\times Y)\times(X\times Y)$ exist as well (in $\mathcal{C}$ and $\mathcal{D}$) respectively, and that they are both the subobject $\{(x_1,y_1,x_2,y_2) : X\times Y\times X\times Y \mid \varphi(x_1,x_2)\wedge\psi(y_1,y_2)\}$, hence they are equal. In particular, as both the constructions exist and define pseudo equivalence relations in $\mathcal{C}$, we conclude that they would define the same equivalence relation over $(X\times Y)\times(X\times Y)$.

The reason why in $\mathrm{Hex}(\mathcal{C})$ we compute the pullback through a formally different procedure is that here the objects are required to be homotopy equivalence relation, that is, pseudo equivalence relations represented by fibrations.

The following couple of results are needed in order to conclude that $\mathrm{Hex}(\mathcal{C})$ is exact, whenever $\mathcal{C}$ is a path category.

**Lemma 1.10.** *If $\mathcal{C}$ is a path category, then, up to precomposing by an isomorphism of* $\mathrm{Hex}(\mathcal{C})$*, every arrow of* $\mathrm{Hex}(\mathcal{C})$ *is represented by a fibration of* $\mathcal{C}$*.*

*Proof.* Let $[f]$ be an arrow $(X, R) \to (Y, S)$, being $R$ and $S$ homotopy equivalence relations. Let us factor $f$ as $X \xrightarrow{w_f} P_f \xrightarrow{p_f} Y$, where $w_f$ is a section of an acyclic fibration $l$ and $p_f$ is a fibration (see Proposition A.5). Let us consider the diagram:

$$\begin{tikzcd} R \arrow[d] \arrow[rr, bend left, "1_R"] & (l\times l)^*R \arrow[r] \arrow[d] & R \arrow[d] \\ X\times X \arrow[r, "w_f\times w_f"] & X_f\times X_f \arrow[r, "l\times l"] & X\times X \end{tikzcd}$$

which commutes, hence (by the universal property of $(l \times l)^*R$) there is unique an arrow $R \to (l \times l)^*R$ making it commute and $w_f$ represents an arrow $(X, R) \to (X_f, (l \times l)^*R)$. Now, we know that $lw_f = 1_X$. Moreover, as $1_{X_f} \simeq w_f l$, there is an arrow $X_f \to PX_f$ such that $(X_f \to PX_f \xrightarrow{\langle s,t\rangle} X_f \times X_f) = \langle 1_{X_f}, w_f l\rangle$. By Proposition 1.7 the homotopy equivalence relation $PX_f \xrightarrow{\langle s,t\rangle} X_f \times X_f$ factors through the homotopy equivalence relation $(l \times l)^*R \to X_f \times X_f$, that is, there is an arrow $PX_f \to (l \times l)^*R$ such that $(PX_f \to (l \times l)^*R \to X_f \times X_f) = \langle s, t\rangle$. In particular the arrow $(X_f \to (l \times l)^*R) := (X_f \xrightarrow{r} PX_f \to (l \times l)^*R)$ is such that $(X_f \to (l \times l)^*R \to X_f \times X_f) = \langle 1_{X_f}, w_f l\rangle$. We proved that $[w_f][l] = [1_{X_f}]$, hence $[w_f]$ is an isomorphism.

Observe that the arrow $R \to (l \times l)^*R$ is a weak equivalence by 4. of Definition 1.3 and being $(l \times l)^*R \to R$ a weak equivalence, as it is the pullback of a weak equivalence along a fibration (see Proposition A.6). Moreover, the following diagram:

$$\begin{tikzcd} R \arrow[d] \arrow[rr, "f'"] & & S \arrow[d] \\ (i\times i)^*R \arrow[r] & X\times X \arrow[r, "p_f\times p_f"] & Y\times Y \end{tikzcd}$$

commutes up to homotopy and the bottom arrow is a fibration. Hence we can replace the weak equivalence $R \to (i \times i)^*R'$ with an homotopic one such that the diagram commutes (see Theorem A.11). Hence, being $S \to Y \times Y$ a fibration and by Theorem A.12, there is an arrow $(i \times i)^*R' \to S$ such that the following:

$$\begin{tikzcd} (i\times i)^*R \arrow[r] \arrow[d] & S \arrow[d] \\ X\times X \arrow[r, "p_f\times p_f"] & Y\times Y \end{tikzcd}$$

commutes and $p_f$ represents an arrow $(X, (i \times i)^*R) \to (Y, S)$. As $[p_f][w_f] = [f]$ we are done. Q.E.D.

**Remark 1.11.** Let $\mathcal{C}$ be a path category and let $(X, R)$ and $(Y, S)$ be objects of $\mathrm{Hex}(\mathcal{C})$. Then their product is the pullback of the unique arrows $(X, R) \to (1, \langle 1_1, 1_1\rangle)$ and $(Y, S) \to (1, \langle 1_1, 1_1\rangle)$. Let us compute it. The pullback of $\langle 1_1, 1_1\rangle$ along $X \times Y \to 1 \times 1$ is the identity arrow over $X \times Y$, because $\langle 1_1, 1_1\rangle$ is an isomorphism. Moreover the pullback of the arrow $(\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \to (X \times Y) \times (X \times Y)$ along the identity over $X \times Y$ is the arrow $(\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \to (X \times Y) \times (X \times Y)$ itself. Hence the product of $(X, R)$ and $(Y, S)$ in $\mathrm{Hex}(\mathcal{C})$ is the object:

$$(X \times Y, (\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \to (X \times Y) \times (X \times Y))$$

together with the arrows $(X \times Y, (\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \to (X \times Y) \times (X \times Y)) \to (X, R)$ and $(X \times Y, (\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \to (X \times Y) \times (X \times Y)) \to (Y, S)$ represented by the arrows $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$ respectively.

Let us assume that $f_1$ and $f_2$ represent arrows $(Z, T) \to (X, S)$ and $(Z, T) \to (Y, S)$ respectively. Observe that the diagram:

$$\begin{tikzcd}
T \arrow[d] \arrow[rr, bend left, "f_1'"] & (\pi_1 \times \pi_1)^* R \arrow[r] \arrow[d] & R \arrow[d] \\
Z \times Z \arrow[r, "\langle f_1, f_2\rangle \times \langle f_1, f_2\rangle"] & (X \times Y) \times (X \times Y) \arrow[r, "\pi_1 \times \pi_1"] & X \times X
\end{tikzcd}$$

commutes, being $f_1'$ an arrow witnessing that $f_1$ preserves the equivalence relation. Hence there is unique an arrow $T \to (\pi_1 \times \pi_1)^* R$ making the diagram commute. Analogously there is an arrow $T \to (\pi_1 \times \pi_1)^* R$ making the diagram:

$$\begin{tikzcd}
T \arrow[d] & (\pi_2 \times \pi_2)^* S \arrow[d] \\
Z \times Z \arrow[r, "\langle f_1, f_2\rangle \times \langle f_1, f_2\rangle"] & (X \times Y) \times (X \times Y)
\end{tikzcd}$$

commute. Moreover, since the equality:

$$(T \to (\pi_1 \times \pi_1)^* R \to (X \times Y) \times (X \times Y)) = (T \to (\pi_2 \times \pi_2)^* S \to (X \times Y) \times (X \times Y))$$

holds (both the sides equal the arrow $T \to Z \times Z \xrightarrow{\langle f_1, f_2\rangle \times \langle f_1, f_2\rangle} (X \times Y) \times (X \times Y)$) and since the following square:

$$\begin{tikzcd}
(\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \arrow[r] \arrow[d] & (\pi_2 \times \pi_2)^* S \arrow[d] \\
(\pi_1 \times \pi_1)^* R \arrow[r] & (X \times Y) \times (X \times Y)
\end{tikzcd}$$

is a pullback, there is unique an arrow $T \to (\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S$ that, postcomposed by the arrows $(\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \to (\pi_1 \times \pi_1)^* R$ and $(\pi_1 \times \pi_1)^* R \cap (\pi_2 \times \pi_2)^* S \to$

$(\pi_2 \times \pi_2)^*S$ respectively, yelds the arrows $T \to (\pi_1 \times \pi_1)^*R$ and $T \to (\pi_2 \times \pi_2)^*S$ of before respectively. Therefore the diagram:

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & (\pi_1 \times \pi_1)^*R \cap (\pi_2 \times \pi_2)^*S \arrow[d] \\
Z \times Z \arrow[r, "\langle f_1, f_2\rangle \times \langle f_1, f_2\rangle"] & (X \times Y) \times (X \times Y)
\end{tikzcd}$$

commutes and $\langle f_1, f_2\rangle$ represents an arrow $(Z, T) \to (X \times Y, (\pi_1 \times \pi_1)^*R \cap (\pi_2 \times \pi_2)^*S)$. Finally, as $[\pi_1][\langle f_1, f_2\rangle] = [\pi_1\langle f_1, f_2\rangle] = [f_1]$ and $[\pi_2][\langle f_1, f_2\rangle] = [\pi_2\langle f_1, f_2\rangle] = [f_2]$, we conclude that:

$$[\langle f_1, f_2\rangle] = \langle [f_1], [f_2]\rangle.$$

**Proposition 1.12.** *Let $\mathcal{C}$ be a path category. Then* $\mathrm{Hex}(\mathcal{C})$ *is exact.*

*Proof.* We need to prove that every equivalence relation is a kernel pair. By Lemma 1.10, up to precomposing by an isomorphism of $\mathrm{Hex}(\mathcal{C})$, we can assume that an equivalence relation of $\mathrm{Hex}(\mathcal{C})$ is represented by a fibration of $\mathcal{C}$, hence it is of the form:

$$(Y, S) \xrightarrow{[f]=[\langle f_1, f_2\rangle]=\langle [f_1],[f_2]\rangle} (X \times X, (\pi_1 \times \pi_1)^*R \cap (\pi_2 \times \pi_2)^*R)$$

for some fibration $Y \xrightarrow{f=\langle f_1, f_2\rangle} X \times X$ of $\mathcal{C}$ and some homotopy equivalence relations $S$ over $Y$ and $R$ over $X$ (here we used Remark 1.11). Because of the form of the regular epi-mono factorization of an arrow in $\mathrm{Hex}(\mathcal{C})$ (see the proof of Proposition 1.8), up to precomposing by an isomorphism, every monomorphism of $\mathrm{Hex}(\mathcal{C})$ is of the form $(A, (\alpha \times \alpha)^*T) \xrightarrow{\alpha} (B, T)$ for some arrow $A \xrightarrow{\alpha} B$ of $\mathcal{C}$. As every equivalence relation is a monomorphism, without loss of generality, we can assume that:

$$S = (f \times f)^*((\pi_1 \times \pi_1)^*R \cap (\pi_2 \times \pi_2)^*R).$$

Let us consider the following diagram:

$$\begin{tikzcd}
R \times_X Y \times_X R \arrow[r, equal] & R \times_X Y \times_Y Y \times_X R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \times_X R \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "r_1"] \\
 & R \times_X Y \arrow[r, "q_2"] \arrow[d, "q_1"'] & Y \arrow[r, "f_2"'] \arrow[d, "f_1"] & X \\
 & R \arrow[r, "r_2"'] & X &
\end{tikzcd}$$

whose squares are pullbacks, and let us observe that an $I$-generalised element $I \xrightarrow{i} R \times_X Y \times_X R$, for some object $I$ of $\mathcal{C}$ is given by a triple $(I \xrightarrow{r} R, I \xrightarrow{y} Y, I \xrightarrow{r'} R)$ such that $r_2 r = f_1 y$ and $f_2 y = r_1 r'$, and viceversa. Indeed, the assignment:

$$r := q_1\pi_1 i,\ y := q_2\pi_1 i = p_1\pi_2 i \text{ and } r' := p_2\pi_2 i$$

defines the claimed bijection (which is indeed a bijection because of the universal properties of the given pullbacks). Hence, for every object $I$ of $\mathcal{C}$, we can consider the map of sets:

$$\begin{aligned}
\mathcal{C}(I, R \times_X Y \times_X R) &\xrightarrow{\varphi_I} \mathcal{C}(I, X \times X) \\
\langle r, y, r'\rangle = \langle\langle r, y\rangle, \langle y, r'\rangle\rangle &\longmapsto \langle r_1 r, r_2 r'\rangle
\end{aligned}$$

which is natural in $I$. Then, by Yoneda's Lemma, there is unique an arrow $R \times_X Y \times_X R \xrightarrow{t} X \times X$ such that, for every object $I$ of $\mathcal{C}$ the map $\varphi_I$ is the postcomposition through $t$. Indeed, according to Remark A.17, the arrow $t$ is *the unique arrow* $R \times_X Y \times_X R \xrightarrow{d} X \times X$ *of* $\mathcal{C}$ *that in a better world would be such that* $d(i) = \varphi_1(i)$ *for every* $i \in R \times_X Y \times_X R$.

According to the assignment (in order to verify it, it suffices to apply the universal properties of the pullbacks), if $I \xrightarrow{i} R \times_X Y \times_X R$ is an $I$-generalized element, then:

$$\begin{aligned} i &= \langle q_1\pi_1 i, q_2\pi_1 i = p_1\pi_2 i, p_2\pi_2 i\rangle \\ &= \langle\langle q_1\pi_1 i, q_2\pi_1 i\rangle, \langle p_1\pi_2 i, p_2\pi_2 i\rangle\rangle \\ &= \langle\langle q_1, q_2\rangle\pi_1 i, \langle p_1, p_2\rangle\pi_2 i\rangle \\ &= \langle\langle q_1, q_2\rangle\pi_1, \langle p_1, p_2\rangle\pi_2\rangle i, \end{aligned}$$

hence it is the case that $\varphi_I(i) = \langle r_1q_1\pi_1 i, r_2p_2\pi_2 i\rangle = \langle r_1q_1\pi_1, r_2p_2\pi_2\rangle i$, that is, $\varphi_I(i)$ is the postcomposition through $\langle r_1q_1\pi_1, r_2p_2\pi_2\rangle$, which is a homotopy equivalence relation of $\mathcal{C}$. This implies that $t = \langle r_1q_1\pi_1, r_2p_2\pi_2\rangle$. Hence $(X, t)$ is an object of $\mathrm{Hex}(\mathcal{C})$ and $1_X$ represents an arrow $(X, R) \to (X, t)$.

It is the case that the arrow $(X, R) \to (X, t)$ coequalizes $[f_1]$ and $[f_2]$, as clearly there is an arrow $Y \xrightarrow{h} R \times_X Y \times_X R$ of $\mathcal{C}$ such that $\langle f_1, f_2\rangle = th$. Indeed $h$ is *the unique arrow* $Y \xrightarrow{k} R \times_X Y \times_X R$ *of* $\mathcal{C}$ *that in a better world would be such that* $k(y) = \psi_1(y)$ *for every* $y \in Y$ (see Remark A.17), being $\psi$ the natural transformation $\mathcal{C}(-, Y) \to \mathcal{C}(-, R \times_X Y \times_X R)$ such that:

$$\begin{aligned} \mathcal{C}(I, Y) &\xrightarrow{\psi_I} \mathcal{C}(I, R \times_X Y \times_X R) \\ y &\longmapsto \langle h_1 y, y, h_2 y\rangle = \langle\langle h_1 y, y\rangle, \langle y, h_2 y\rangle\rangle \end{aligned}$$

for every object $I$ of $\mathcal{C}$, being $h_1$ and $h_2$ fixed arrows $Y \to R$ such that $\langle f_1, f_1\rangle = \langle r_1, r_2\rangle h$ and $\langle f_2, f_2\rangle = \langle r_1, r_2\rangle h$ (they exist, since $R$ is a pseudo equivalence relation over $X$). Indeed, we observe that $r_2(h_1 y) = (r_2 h_1)y = f_1 y$ and $f_2 y = (r_1 h_2)y = r_1(h_2 y)$, hence $\langle h_1 y, y, h_2 y\rangle$ is actually an $I$-generalized element of $R \times_X Y \times_X R$, for every $I$-generalized element $y$ of $Y$. Again, as for every $y \in \mathcal{C}(I, Y)$ it is the case that:

$$\psi_I(y) = \langle\langle h_1 y, y\rangle, \langle y, h_2 y\rangle\rangle = \langle\langle h_1, 1_Y\rangle, \langle 1_Y, h_2\rangle\rangle y,$$

we deduce that $\psi_I$ is the postcomposition by $\langle\langle h_1, 1_Y\rangle, \langle 1_Y, h_2\rangle\rangle$ for every object $I$ of $\mathcal{C}$, hence we must admit that $h = \langle\langle h_1, 1_Y\rangle, \langle 1_Y, h_2\rangle\rangle$. Finally, we observe that:

$$\begin{aligned} th &= \langle r_1q_1\pi_1, r_2p_2\pi_2\rangle\langle\langle h_1, 1_Y\rangle, \langle 1_Y, h_2\rangle\rangle \\ &= \langle r_1q_1\pi_1\langle\langle h_1, 1_Y\rangle, \langle 1_Y, h_2\rangle\rangle, r_2p_2\pi_2\langle\langle h_1, 1_Y\rangle, \langle 1_Y, h_2\rangle\rangle\rangle \\ &= \langle r_1q_1\langle h_1, 1_Y\rangle, r_2p_2\langle 1_Y, h_2\rangle\rangle \\ &= \langle r_1h_1, r_2h_2\rangle \\ &= \langle f_1, f_2\rangle \end{aligned}$$

as we claimed. Finally applying the usual characterization of the pullback in $\mathrm{Hex}(\mathcal{C})$, we could verify that the couple $([f_1], [f_2])$ is the kernel pair of $(X, R) \to (X, t)$, hence $\mathrm{Hex}(\mathcal{C})$ is exact. Q.E.D.

We illustrate two alternative presentations of the exact completion of a given path category. The first one is a natural variation of the definition:

Let $\mathcal{C}$ be a path category and let $\mathrm{Hex}'(\mathcal{C})$. We basically define the category $\mathrm{Hex}'(\mathcal{C})$ as the $\mathrm{Ex}(\mathcal{C})$ would be if $\mathcal{C}$ was finitely complete. In other words, $\mathrm{Hex}'(\mathcal{C})$ is defined as follows:

1. An object of $\mathrm{Hex}'(\mathcal{C})$ is a couple $(X, R \xrightarrow{\langle r_1, r_2\rangle} X \times X)$, being $\langle r_1, r_2\rangle$ a pseudo equivalence relation of $\mathcal{C}$.

2. Whenever $(X, \langle r_1, r_2\rangle)$ and $(Y, \langle s_1, s_2\rangle)$ are couples as in 1., let us consider the arrows $X \xrightarrow{f} Y$ of $\mathcal{C}$ such that there is an arrow $R \xrightarrow{f'} S$ such that $(f \times f)\langle r_1, r_2\rangle \simeq \langle s_1, s_2\rangle f'$. We say that two parallel arrows $f$ and $g$ satisfying this property are equivalent if and only if there is an arrow $X \to S$ such that $(X \to S \xrightarrow{\langle s_1, s_2\rangle} Y \times Y) \simeq \langle f, g\rangle$. An arrow of $\mathrm{Hex}'(\mathcal{C})$ with source $(X, \langle r_1, r_2\rangle)$ and target $(Y, \langle s_1, s_2\rangle)$ is an equivalence class (modulo this equivalence relation) of arrows $X \xrightarrow{f} Y$ of $\mathcal{C}$ satisfying the property of before.

3. The equivalence relation defined in 2. is a congruence. Then the composition is well-defined if we stipulate that the composition of two arrows of $\mathrm{Hex}'(\mathcal{C})$ is represented by the composition in $\mathcal{C}$ of two representatives of them.

Then the following holds:

**Theorem 1.13.** *Let $\mathcal{C}$ be a path category. Then there is an equivalence of categories* $\mathrm{Hex}(\mathcal{C}) \simeq \mathrm{Hex}'(\mathcal{C})$.

*Proof.* Of course every homotopy equivalence relation is a pseudo equivalence relation and the arrows of $\mathrm{Hex}(\mathcal{C})$ between two homotopy equivalence relations are precisely the arrows between them seen as pseudo equivalence relations. Hence the inclusion $\mathrm{Hex}(\mathcal{C}) \hookrightarrow \mathrm{Hex}'(\mathcal{C})$ is fully faithful. Moreover, if $R \xrightarrow{\langle r_1, r_2\rangle} X \times X$ is a pseudo equivalence relation and $(R \xrightarrow{w} R' \xrightarrow{\langle r_1', r_2'\rangle} X \times X)$ is his weak-fibre factorization, then $R' \xrightarrow{\langle r_1', r_2'\rangle} X \times X$ is a pseudo equivalence relation as well, hence it is a homotopy equivalence relation. Moreover $1_X$ together with $w$ represents an arrow $(X, \langle r_1, r_2\rangle) \to (X, \langle r_1', r_2'\rangle)$ of $\mathrm{Hex}'(\mathcal{C})$ and $1_X$ together with a pseudo inverse of $w$ represents an arrow $(X, \langle r_1', r_2'\rangle) \to (X, \langle r_1, r_2\rangle)$. Of course their compositions are the unique arrows $(X, \langle r_1, r_2\rangle) \to (X, \langle r_1, r_2\rangle)$ and $(X, \langle r_1', r_2'\rangle) \to (X, \langle r_1', r_2'\rangle)$ represented by $1_X$. Both of them happen to be the identities of $(X, \langle r_1, r_2\rangle)$ and $(X, \langle r_1', r_2'\rangle)$. Hence there is an isomorphism of $\mathrm{Hex}'(\mathcal{C})$ between $(X, \langle r_1, r_2\rangle)$ and $(X, \langle r_1', r_2'\rangle)$. Therefore $\mathrm{Hex}(\mathcal{C}) \hookrightarrow \mathrm{Hex}'(\mathcal{C})$ is essentially surjective. Q.E.D.

A third presentation of the exact completion of a given path category proceeds as follows. Let $\mathcal{C}$ be a path category and let $\mathrm{Ex}'(\mathcal{C})$ be the category defined as follows:

1. An object of $\mathrm{Ex}'(\mathcal{C})$ is a couple $(X, R \xrightarrow{\langle r_1, r_2\rangle} X \times X)$, being $\langle r_1, r_2\rangle$ a homotopy equivalence relation of $\mathcal{C}$.

2. Whenever $(X, \langle r_1, r_2\rangle)$ and $(Y, \langle s_1, s_2\rangle)$ are couples as in 1., an arrow of $\mathrm{Ex}'(\mathcal{C})$ with source $(X, \langle r_1, r_2\rangle)$ and target $(Y, \langle s_1, s_2\rangle)$ is just an arrow $X \xrightarrow{f} Y$ of $\mathcal{C}$ such that there is an arrow $R \xrightarrow{f'} S$ of $\mathcal{C}$ such that $(f \times f)\langle r_1, r_2\rangle = \langle s_1, s_2\rangle f'$.

We stipulate that an arrow $(X, \langle r_1, r_2 \rangle) \xrightarrow{f} (Y, \langle s_1, s_2 \rangle)$ of $\mathrm{Ex}'(\mathcal{C})$ is a fibration of $\mathrm{Ex}'(\mathcal{C})$ if and only if $X \xrightarrow{f} Y$ is a fibration of $\mathcal{C}$ and, whenever the following square:

$$\begin{tikzcd} X \times_Y S \arrow[r, "p_2"] \arrow[d, "p_1"'] & S \arrow[d, "s_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pullback, there is an arrow $X \times_Y S \xrightarrow{\nabla} R$ of $\mathcal{C}$ such that $(X \times_Y S \xrightarrow{\nabla} R \xrightarrow{r_1} X) = (X \times_Y S \xrightarrow{p_1} X)$ and $(X \times_Y S \xrightarrow{\nabla} R \xrightarrow{r_2} X \xrightarrow{f} Y) = (X \times_Y S \xrightarrow{p_2} S \xrightarrow{s_2} Y)$.

We say that two parallel arrows $f$ and $g$ of $\mathrm{Ex}'(\mathcal{C})$ of source $(X, \langle r_1, r_2 \rangle)$ and target $(Y, \langle s_1, s_2 \rangle)$ are equivalent (and in this case we write $f \sim g$) if and only if there is an arrow $X \to S$ such that $(X \to S \xrightarrow{\langle s_1, s_2 \rangle} Y \times Y) = (X \xrightarrow{\langle f, g \rangle} Y \times Y)$.

We stipulate that an arrow $(X, \langle r_1, r_2 \rangle) \xrightarrow{f} (Y, \langle s_1, s_2 \rangle)$ is a weak equivalence of $\mathrm{Ex}'(\mathcal{C})$ if and only if there is an arrow $(Y, \langle s_1, s_2 \rangle) \xrightarrow{g} (X, \langle r_1, r_2 \rangle)$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

Then the following holds:

**Proposition 1.14.** *Let $\mathcal{C}$ be a path category. Then $\mathrm{Ex}'(\mathcal{C})$ is a path category (with the notions of fibration and weak equivalence introduced above). Moreover $\mathrm{Ho}(\mathrm{Ex}'(\mathcal{C})) = \mathrm{Hex}(\mathcal{C})$ (see Appendix A.1).*

*Sketch of proof.* At first one verifies that the conditions of Definition 1.3 are verified. Then one observes that any two parallel arrows of $\mathrm{Ex}'(\mathcal{C})$ are $\sim$-equivalent if and only if they are homotopic. Since $\mathrm{Ho}(\mathrm{Ex}'(\mathcal{C}))$ is the category whose objects are the ones of $\mathrm{Ex}'(\mathcal{C})$ and whose arrows are the equivalence classes of the ones of $\mathrm{Ex}'(\mathcal{C})$ modulo the notion of homotopy of $\mathrm{Ex}'(\mathcal{C})$, it is the case that an arrow of $\mathrm{Ho}(\mathrm{Ex}'(\mathcal{C}))$ is precisely an equivalence class modulo the relation $\sim$ of arrows of $\mathrm{Ex}'(\mathcal{C})$. But this is precisely an arrow of $\mathrm{Hex}(\mathcal{C})$. Q.E.D.

## 1.3 About the Exact Categories Obtained through these Procedures

In this last section we show a characterisation of the exact categories obtained through the procedure presented in Section 1.1. After that, we present the analogous results for the exact categories obtained through the procedure presented in Section 1.2.

We start from the following:

**Lemma 1.15.** *Let $\mathcal{C}$ be a finitely complete category. Then the objects of the image of the embedding $\Gamma \colon \mathcal{C} \to \mathrm{Ex}(\mathcal{C})$ are projective.*

*Proof.* Let $X$ be an object of $\mathcal{C}$. We know by the proof of Theorem 1.2 that (up to postcomposing by an isomorphism of $\mathrm{Ex}(\mathcal{C})$) the regular epimorphisms of $\mathrm{Ex}(\mathcal{C})$ are the arrows represented by the identities of $\mathcal{C}$. Let us consider such a regular epimorphism $(Y, \langle r_1, r_2 \rangle) \xrightarrow{[1_Y]} (Y, \langle s_1, s_2 \rangle)$ and let us assume that there is an arrow $(X, \langle 1_X, 1_X \rangle) \xrightarrow{[f]} (Y, \langle s_1, s_2 \rangle)$. As $\langle r_1, r_2 \rangle$ is a pseudo-equivalence relation, there is an arrow $X \xrightarrow{h} R$ such that

$\langle r_1, r_2\rangle h = \langle f, f\rangle$, that is, the following diagram:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\langle 1_X, 1_X\rangle"'] & R \arrow[d, "\langle r_1, r_2\rangle"] \\ X \times X \arrow[r, "f\times f"] & Y \times Y \end{tikzcd}$$

commutes in $\mathcal{C}$ and $f$ represents an arrow $(X, \langle 1_X, 1_X\rangle) \to (Y, \langle r_1, r_2\rangle)$. Moreover the diagram:

$$\begin{tikzcd} (X, \langle 1_X, 1_X\rangle) \arrow[r, "{[f]}"] \arrow[dr, "{[f]}"'] & (Y, \langle s_1, s_2\rangle) \\ & (Y, \langle r_1, r_2\rangle) \arrow[u, "{[1_Y]}"'] \end{tikzcd}$$

clearly commutes in $\mathrm{Ex}(\mathcal{C})$ and $(X, \langle 1_X, 1_X\rangle)$ is a projective object of $\mathrm{Ex}(\mathcal{C})$. Q.E.D.

We can state and prove the following:

**Proposition 1.16.** *Let $\mathcal{C}$ be a finitely complete category. Then:*

1. *The image of $\mathcal{C}$ in $\mathrm{Ex}(\mathcal{C})$ through the embedding $\Gamma\colon \mathcal{C} \to \mathrm{Ex}(\mathcal{C})$ is finitely complete;*
2. *The projective objects of $\mathrm{Ex}(\mathcal{C})$ are precisely the objects of the essential image of $\Gamma$;*
3. *The category $\mathrm{Ex}(\mathcal{C})$ has enough projectives.*

*Proof.*
*1.* This follows since $\mathcal{C}$ is finitely complete and $\Gamma$ preserves finite limits.

*2.* By Lemma 1.15 it suffices to prove that a given projective object $(X, R)$ is isomorphic to an object of the image of $\Gamma$. By the proof of Lemma 1.15, the identity $1_X$ represents a regular epimorphism $(X, \langle 1_X, 1_X\rangle) \xrightarrow{c} (X, R)$. As $(X, R)$ is projective, there is an arrow $X \xrightarrow{s} X$ of $\mathcal{C}$, representing an arrow $(X, R) \to (X, \langle 1_X, 1_X\rangle)$ such that the diagram:

$$\begin{tikzcd} (X, \langle 1_X, 1_X\rangle) \arrow[r, "c"] & (X, R) \\ & (X, R) \arrow[u, "{[1_X]}"'] \arrow[ul, "{[s]}"] \end{tikzcd}$$

commutes. Let $(E, S) \xrightarrow{[e]} (X, \langle 1_X, 1_X\rangle)$ be the equaliser of the couple $([s]c, [1_X])$ of arrows $(X, \langle 1_X, 1_X\rangle) \to (X, \langle 1_X, 1_X\rangle)$. Then $[e]$ is a monomorphism of $\mathrm{Ex}(\mathcal{C})$, hence, up to precomposing by an isomorphism, we can assume (see the proof of 1.2) that:

$$S = (e \times e)^* \langle 1_X, 1_X\rangle = \langle e^* 1_X, e^* 1_X\rangle = \langle 1_E, 1_E\rangle.$$

Observe that the arrow $[s]$ equalises the pair $([s]c, [1_X])$. Hence, by the universal property of the arrow $[e]$, there is unique an arrow $(X, R) \xrightarrow{r} (E, \langle 1_E, 1_E\rangle)$ such that $[e]r = [s]$. We are done if $r$ is an isomorphism. This is indeed true, since:

- it is the case that $(c[e])r = c[s] = [1_X] = 1_{(X,R)}$;
- it is the case that $[e]r(c[e]) = [s]c[e] = [e]$, hence $r(c[e]) = 1_{(E,\langle 1_E, 1_E\rangle)}$;

that is, $c[e]$ is the inverse of $r$.

*3.* Let $(X, R)$ be an object of $\mathrm{Ex}(\mathcal{C})$. Being $R$ a pseudo-equivalence relation, there is an arrow $X \xrightarrow{\rho} R$ such that the diagram:

$$\begin{tikzcd} X \arrow[r, "\langle 1_X, 1_X \rangle"] \arrow[dr, "\rho"'] & X \times X \\ & R \arrow[u] \end{tikzcd}$$

commutes, hence $1_X$ represents an arrow $(X, \langle 1_X, 1_X \rangle) \to (X, R)$, which is an epimorphism as it is represented by an identity. Moreover, by the previous lemma $(X, \langle 1_X, 1_X \rangle)$ is a projective object of $\mathrm{Ex}(\mathcal{C})$. Q.E.D.

Viceversa, the following holds:

**Proposition 1.17.** *Let $\mathcal{E}$ be an exact category with enough projectives such that the full subcategory $\mathcal{P}$ of $\mathcal{E}$ spanned by its projective objects is closed under finite limits. Then there is an equivalence of categories* $\mathrm{Ex}(\mathcal{P}) \simeq \mathcal{E}$.

*Proof.* Let us define a functor $F\colon \mathrm{Ex}(\mathcal{P}) \to \mathcal{E}$. Let $R \xrightarrow{\langle r_0, r_1 \rangle} X \times X$ be an object of $\mathrm{Ex}(\mathcal{P})$, that is, a pseudo equivalence relation in $\mathcal{P}$. Let:

$$(R \xrightarrow{e} R' \xrightarrow{\langle r'_0, r'_1 \rangle} X \times X)$$

be its regular-epi mono factorization in $\mathcal{E}$. Then the image $R' \xrightarrow{\langle r'_0, r'_1 \rangle} X \times X$ is an equivalence relation of $\mathcal{E}$, that is, a kernel pair. Let $X \xrightarrow{q} X/R$ be its quotient, which is also the quotient of $\langle r_0, r_1 \rangle$, being $e$ an epimorphism. Then we stipulate that $F\langle r_0, r_1 \rangle$ is $X/R$.

Now, let $[f]$ be an arrow $(R \xrightarrow{\langle r_0, r_1 \rangle} X \times X) \to (S \xrightarrow{\langle s_0, s_1 \rangle} Y \times Y)$ of $\mathrm{Ex}(\mathcal{P})$. Let us consider the commutative diagram:

$$\begin{tikzcd} R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d, "f'"'] & X \arrow[r, "q"] \arrow[d, "f"] & F\langle r_0, r_1 \rangle \\ S \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & Y \arrow[r, "q'"] & F\langle s_0, s_1 \rangle \end{tikzcd}$$

and observe that $q'f$ coequalizes $r_0$ and $r_1$. Being $q$ the initial arrow coequalizing $r_0$ and $r_1$, there is unique an arrow $F\langle r_0, r_1 \rangle \to F\langle s_0, s_1 \rangle$ making the diagram commute. We stipulate that $F[f]$ is this arrow. Let us assume that $X \xrightarrow{g} Y$ is an other representative of $[f]$, i.e. $[f] = [g]$. Then there is an arrow $X \xrightarrow{h} S$ such that:

$$(X \xrightarrow{h} S \xrightarrow{\langle s_0, s_1 \rangle} Y \times Y) = (X \xrightarrow{\langle f, g \rangle} Y \times Y).$$

Then we observe that $q'g = q's_0h = q's_1h = q'f$, hence it is the case that $(F[g])q = q'g = q'f$. Hence, being $(F[f])$ the unique arrow $F\langle r_0, r_1 \rangle \to F\langle s_0, s_1 \rangle$ such that $(F[f])q = q'f$, it is the case that $F[f] = F[g]$. We conclude that $F$ is well-defined and it is a functor because of the universal property of the coequalizers. We are going to prove that it is an equivalence of categories.

*Essential surjectivity.* Let $E$ be an object of $\mathcal{E}$. Since $\mathcal{E}$ has enough projectives, there is a projective object $X$ of $\mathcal{E}$ and a regular epimorphism $X \xrightarrow{q} E$. Hence $e$ is the coequalizer of its own kernel pair $R \xrightarrow{\langle r_0, r_1 \rangle} X \times X$ by Lemma A.14. Again, let $P$ be a projective object of $\mathcal{E}$ and let $P \to R$ be a regular epimorphism. Then $P \to R \xrightarrow{\langle r_0, r_1 \rangle} X \times X$ is an object of $\mathrm{Ex}(\mathcal{P})$ such that $F\langle r_0, r_1 \rangle = E$.

*Faithfulness.* Let $\langle r_0, r_1 \rangle$ and $\langle s_0, s_1 \rangle$ be objects of $\mathrm{Ex}(\mathcal{P})$ and let $[f], [g]$ be arrows $\langle r_0, r_1 \rangle \to \langle s_0, s_1 \rangle$ of $\mathrm{Ex}(\mathcal{P})$. Moreover, let us assume that $F[f] = F[g]$. Let us consider the diagram:

$$\begin{tikzcd}
R \arrow[r, "e"] \arrow[d, shift right, "f'"'] \arrow[d, shift left, "g'"] & R' \arrow[r, shift left, "r_0'"] \arrow[r, shift right, "r_1'"'] & X \arrow[r, "q"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & F\langle r_0, r_1 \rangle \arrow[d, "{F[f]=F[g]}"] \\
S \arrow[r, "e'"] & S' \arrow[r, shift left, "s_0'"] \arrow[r, shift right, "s_1'"'] & Y \arrow[r, "q'"] & F\langle s_0, s_1 \rangle
\end{tikzcd}$$

and observe that $q'$ coequalizes $f$ and $g$. As the couple $(s_0', s_1')$ is the kernel pair of $q'$ by Lemma A.15, there is unique an arrow $X \to S'$ such that:

$$(X \to S' \xrightarrow{\langle s_0', s_1' \rangle} Y \times Y) = (X \xrightarrow{\langle f, g \rangle} Y \times Y).$$

Moreover, as $X$ is projective and $e'$ is a regular epimorphism, there is an arrow $X \xrightarrow{h} S$ such that $(X \xrightarrow{h} S \xrightarrow{e'} S') = (X \to S')$, hence $\langle s_0, s_1 \rangle h = \langle f, g \rangle$, i.e. $[f] = [g]$.

*Fullness.* Let $l$ be an arrow $F\langle r_0, r_1 \rangle \to F\langle s_0, s_1 \rangle$ of $\mathcal{E}$. Being $X$ projective, there is an arrow $X \xrightarrow{f} Y$ such that the square:

$$\begin{tikzcd}
R \arrow[r, "e"] & R' \arrow[r, shift left, "r_0'"] \arrow[r, shift right, "r_1'"'] & X \arrow[r, "q"] \arrow[d, "f"] & F\langle r_0, r_1 \rangle \arrow[d, "l"] \\
S \arrow[r, "e'"] & S' \arrow[r, shift left, "s_0'"] \arrow[r, shift right, "s_1'"'] & Y \arrow[r, "q'"] & F\langle s_0, s_1 \rangle
\end{tikzcd}$$

commutes. Let $\langle x_0, x_1 \rangle$ be an arrow $I \to X \times X$ of $\mathcal{P}$ such that there is an arrow $I \xrightarrow{r} R$ such that $\langle r_0, r_1 \rangle r = \langle x_0, x_1 \rangle$. Then it is the case that $qx_0 = qx_1$, hence $q'(fx_0) = q'(fx_1)$. Being $(s_0', s_1')$ the kernel pair of $q'$ by Lemma A.15, there is unique an arrow $I \to S'$ such that:

$$(I \to S' \xrightarrow{\langle s_0', s_1' \rangle} Y \times Y) = \langle fx_0, fx_1 \rangle.$$

Moreover, as $I$ is projective and $e'$ is a regular epimorphism, there is an arrow $I \xrightarrow{s} S$ such that $e's = (I \to S)$. Therefore it is the case that $\langle fx_0, fx_1 \rangle = \langle s_0, s_1 \rangle s$. We conclude that $f$ preserves the pseudo equivalence relation, hence the arrow $\langle r_0, r_1 \rangle \xrightarrow{[f]} \langle s_0, s_1 \rangle$ exists and $F[f] = l$. Q.E.D.

Hence, by Proposition 1.16 and Proposition 1.17 we get the following:

**Corollary 1.18.** *Let $\mathcal{D}$ be an exact category and let $\mathcal{D}'$ be a finitely complete full subcategory of $\mathcal{D}$. Then $\mathrm{Ex}(\mathcal{D}') \simeq \mathcal{D}$ if and only if $\mathcal{D}'$ is (up to equivalence) the full subcategory of the projective objects of $\mathcal{D}$ and $\mathcal{D}$ has enough projectives.*

The aim of this section is to determine which of the previous properties valid for the exact completion $\mathrm{Ex}(\mathcal{C})$ of a fintely complete category $\mathcal{C}$ are still true for the exact completion $\mathrm{Hex}(\mathcal{D})$ of a path category $\mathcal{D}$.

The natural way of generalising the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Ex}(\mathcal{C})$ to the general case of a path category $\mathcal{D}$ is the following: we stipulate that every object $X$ of $\mathcal{D}$ is sent to the couple $(X, PX)$. Indeed, if $\mathcal{D}$ is a finitely complete category with the usual path categorical structure, then, for very object $X$ of $\mathcal{D}$, it is the case that:

$$(PX \xrightarrow{\langle s,t\rangle} X \times X) = (X \xrightarrow{\langle 1_X, 1_X\rangle} X \times X)$$

(see Example 1.5). Moreover we stipulate that every arrow of $\mathcal{D}$ is sent to the arrow in $\mathrm{Hex}(\mathcal{C})$ represented by itself (it agrees with the homotopy equivalence relation-structure by Theorem A.9). This defines a functor $\Gamma\colon \mathcal{D} \to \mathrm{Hex}(\mathcal{D})$ which, assuming that $\mathcal{D}$ is a finitely complete category with the usual path categorical structure, coincides with the embedding of Proposition 1.2 and Lemma 1.16. However:

**Remark 1.19.** Let $\mathcal{D}$ be a path category. In general $\Gamma$ is not an embedding (as it happens for instance when $\mathcal{D}$ is a finitely complete category with the usual path categorical structure): it is clearly full, but, if $f, g$ are arrows $X \to Y$ of $\mathcal{C}$ and there is a homotopy $h\colon f \simeq g$, then it is the case that $\Gamma f = [f] = [g] = \Gamma g$. This proves that the image of $\mathcal{D}$ through $\Gamma$ is isomorphic to $\mathrm{Ho}(\mathcal{D})$. Observe that, when $\mathcal{D}$ is a finitely complete category with the path categorical structure of Example 1.5, it is the case that $\mathcal{D}$ and $\mathrm{Ho}(\mathcal{D})$ are isomorphic (as it should be, since in that case $\Gamma$ is an embedding and its image is then isomorphic to $\mathcal{D}$).

The following remark illustrates why the the homotopy equivalence relation $PX \xrightarrow{\langle s,t\rangle} X \times X$ over $X$ is the natural generalisation, in a general path category, of the (pseudo) equivalence relation $X \xrightarrow{\langle 1_X, 1_X\rangle} X \times X$ over $X$ for a finitely complete category.

**Remark 1.20.** Let $\mathcal{C}$ be a path category and let $X$ be an object of $\mathcal{C}$. Then the homotopy equivalence relation $PX \xrightarrow{\langle s,t\rangle} X \times X$ is *(weakly) universal* in the sense of Proposition 1.7: whenever $Y$ is an object of $\mathcal{C}$, $R$ is a homotopy equivalence relation over $Y$ and $f$ is an arrow $X \to Y$ of $\mathcal{C}$, then there is an arrow $PX \to S$ such that:

$$\begin{tikzcd} PX \arrow[r] \arrow[d, "\langle s,t\rangle"'] & R \arrow[d] \\ X \times X \arrow[r, "f\times f"] & Y \times Y \end{tikzcd}$$

commutes. In particular, if $\mathcal{C}$ is a finitely complete category and its path categorical structure is the one of Example 1.5, then the homotopy equivalence relation are presisely the pseudo equivalence relation and hence the same weakly universal property (which in this case is actually a strong one) is enjoyed by the equivalence relation $X \xrightarrow{\langle 1_X, 1_X\rangle} X \times X$: whenever $Y$ is an object of $\mathcal{C}$, $R$ is a pseudo equivalence relation over $Y$ and $f$ is an arrow $X \to Y$ of $\mathcal{C}$, then there is an arrow $X \to S$ such that:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\langle 1_X,1_X\rangle"'] & R \arrow[d] \\ X \times X \arrow[r, "f\times f"] & Y \times Y \end{tikzcd}$$

commutes, by the reflexivity of $R$.

Let us observe that *2.* and *3.* of Proposition 1.16 still hold in the general case:

**Proposition 1.21.** *Let $\mathcal{C}$ be a path category. Then the projective objects of* $\mathrm{Hex}(\mathcal{C})$ *are precisely the ones of the essential image of* $\Gamma$ *and* $\mathrm{Hex}(\mathcal{C})$ *has enough projectives.*

*Proof.* Let $X$ be an object of $\mathcal{C}$. Let us consider an epimorphism $(Y, R) \xrightarrow{[1_Y]} (Y, S)$ of $\mathrm{Hex}(\mathcal{C})$ and an arrow $(X, PX) \xrightarrow{[f]} (Y, S)$. By Proposition 1.7, there is an arrow $PX \to R$ such that the diagram:

$$\begin{tikzcd} PX \arrow[r] \arrow[d, "\langle s,t\rangle"'] & R \arrow[d] \\ X \times X \arrow[r, "f\times f"] & Y \times Y \end{tikzcd}$$

commutes. Hence $f$ represents an arrow $(X, PX) \to (Y, R)$ such that the diagram:

$$\begin{tikzcd} (X, PX) \arrow[r, "{[f]}"] \arrow[dr, "{[f]}"'] & (Y, S) \\ & (Y, R) \arrow[u, "{[1_Y]}"'] \end{tikzcd}$$

commutes in $\mathrm{Hex}(\mathcal{C})$. This proves that $(X, PX)$ is projective. Moreover, if $T$ is a homotopy equivalence relation over $X$, again by Proposition 1.7 there is an arrow $PX \to T$ such that:

$$\begin{tikzcd} PX \arrow[r] \arrow[d, "\langle s,t\rangle"'] & T \arrow[d] \\ X \times X \arrow[r, "1_X\times 1_X"] & X \times X \end{tikzcd}$$

commutes. Hence the arrow $1_X$ of $\mathcal{C}$ represents an arrow $(X, PX) \to (X, T)$ in $\mathrm{Hex}(\mathcal{C})$, which is as usual an epimorphism. This proves that $\mathrm{Hex}(\mathcal{C})$ has enough projectives.

Now, let us assume that $(X, R)$ is a projective object of $\mathrm{Hex}(\mathcal{C})$ and let us consider the arrow $(X, PX) \xrightarrow{c} (X, R)$ represented by $1_X$. As $(X, R)$ is projective, there is an arrow $(X, R) \xrightarrow{[s]} (X, PX)$ such that the diagram:

$$\begin{tikzcd} (X, PX) \arrow[r, "c"] & (X, R) \\ & (X, R) \arrow[u, "{[1_X]}"'] \arrow[ul, "{[s]}"] \end{tikzcd}$$

commutes. Let $(E, S) \xrightarrow{[e]} (X, PX)$ be the equaliser of the couple $([s]c, [1_X])$ of arrows $(X, PX) \to (X, PX)$. Up to precomposing by an isomorphism of $\mathrm{Hex}(\mathcal{C})$, we can assume that $e$ is a fibration of $\mathcal{C}$ by Lemma 1.10. As $[e]$ is a monomorphism of $\mathrm{Ex}(\mathcal{C})$, up to precomposing by an isomorphism of $\mathrm{Hex}(\mathcal{C})$, we can assume that:

$$S = (e \times e)^* PX$$

hence, by Example 3.2 and Proposition 3.3, it is the case that $S = PE$.

Since that the arrow $[s]$ equalises the pair $([s]c, [1_X])$, by the universal property of the arrow $[e]$, there is unique an arrow $(X, R) \xrightarrow{r} (E, PE)$ such that $[e]r = [s]$. We are done because $r$ is an isomorphism, since $(c[e])r = c[s] = [1_X] = 1_{(X,R)}$ and $[e]r(c[e]) = [s]c[e] = [e]$, which implies that $r(c[e]) = 1_{(E,PE)}$. Q.E.D.

The natural generalisation of *1.* of Proposition 1.16 is the following (see Remark 1.23 to clarify this):

**Proposition 1.22.** *Let* $\mathcal{C}$ *be a path category. Then* $\Gamma\colon \mathcal{C} \to \mathrm{Hex}(\mathcal{C})$ *preserves the terminal object and sends homotopy pullbacks to pullbacks.*

*Proof.* The object $\Gamma 1$ is terminal in $\mathrm{Hex}(\mathcal{C})$ by Proposition 1.7, being 1 a terminal object of $\mathcal{C}$. Let $X \xrightarrow{f} A$ and $Y \xrightarrow{g} A$ be arrows of $\mathcal{C}$ and let us consider the pullback:

$$\begin{tikzcd} X\,{}^h\!\!\times_A Y \arrow[r] \arrow[d, "\langle p_1, p_2\rangle"'] & PA \arrow[d, "\langle s,t\rangle"] \\ X \times Y \arrow[r, "f\times g"] & A \times A \end{tikzcd}$$

which exists because $\langle s, t\rangle$ is a fibration. Hence $fp_1 \simeq gp_2$, that is, the following square:

$$\begin{tikzcd} X\,{}^h\!\!\times_A Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

commutes up to homotopy. By Definition A.7 this square constitutes, up to homotopy, the homotopy pullback of the pair $(f, g)$ in $\mathcal{C}$. Moreover, by Proposition A.22, we know that the corresponding quotient:

$$\begin{tikzcd} X\,{}^h\!\!\times_A Y \arrow[r, "{[p_2]}"] \arrow[d, "{[p_1]}"'] & Y \arrow[d, "{[g]}"] \\ X \arrow[r, "{[f]}"] & A \end{tikzcd}$$

in $\mathrm{Ho}(\mathcal{C})$ is a weak pullback square. By Remark 1.19 and according to the isomorphism of categories:

$$\mathrm{Ho}(\mathcal{C}) \cong \Gamma(\mathcal{C}) \hookrightarrow \mathrm{Hex}(\mathcal{C})$$

in order to prove that the corresponding square in $\mathrm{Hex}(\mathcal{C})$ is a pullback square (see the proof of Proposition 1.8) and hence conclude the proof, it is enough to observe that:

- According to the usual presentation of the pullback in $\mathrm{Hex}(\mathcal{C})$, the domain of the homotopy equivalence relation obtained by the pullback of $[f]$ and $[g]$ in $\mathrm{Hex}(\mathcal{C})$ is precisely $X\,{}^h\!\!\times_A Y$;
- According to the usual presentation of the pullback in $\mathrm{Hex}(\mathcal{C})$, the homotopy equivalence relation obtained by the pullback of $[f]$ and $[g]$ in $\mathrm{Hex}(\mathcal{C})$, whose domain is $X\,{}^h\!\!\times_A Y$ is precisely a path object over $X\,{}^h\!\!\times_A Y$, because it is the result of a finite iteration of pullbacks of path objects and, by Proposition 3.3, pullbacks of path objects are path objects. Q.E.D.

**Remark 1.23.** Let $\mathcal{C}$ be a finitely complete category with the usual path categorical structure of Example 1.5 allowing us to see path categories as a natural generalisation of finitely complete categories. In this case, the homotopy pullback of two arrows of $\mathcal{C}$ is precisely their pullback. In fact, if the square:

$$\begin{tikzcd} X\,{}^h\!\times_A Y \arrow[r, "x"] \arrow[d, "\langle p_1,p_2\rangle"'] & A \arrow[d, "\langle 1_A,1_A\rangle"] \\ X \times Y \arrow[r, "f\times g"] & A \times A \end{tikzcd}$$

is a pullback of $\mathcal{C}$, then the square:

$$\begin{tikzcd} X\,{}^h\!\times_A Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

actually commutes. Moreover, whenever $\alpha$ and $\beta$ are arrows $Z \to X$ and $Z \to Y$ such that $f\alpha = g\beta$, then it is the case that $(f \times g)\langle\alpha, \beta\rangle = \langle 1_A, 1_A\rangle f\alpha$, hence there is unique an arrow $Z \xrightarrow{h} X\,{}^h\!\times_A Y$ such that $\langle p_1, p_2\rangle h = \langle\alpha, \beta\rangle$ and $xh = f\alpha$. But, as the latter is a consequence of the former, it is the case that there is unique an arrow $Z \xrightarrow{h} X\,{}^h\!\times_A Y$ such that $\langle p_1, p_2\rangle h = \langle\alpha, \beta\rangle$. We conclude that $(X\,{}^h\!\times_A Y, p_1, p_2)$ is a pullback of $(f, g)$ in $\mathcal{C}$.

Therefore, in this particular case, we observe that the statement of Proposition 1.22 is indeed equivalent to *1.* of Proposition 1.16. In other words, Proposition 1.22 is actually a generalisation of *1.* of Proposition 1.16, as we claimed.

We summarise what we know in the following:

**Corollary 1.24.** *Let $\mathcal{C}$ be a path category. Then:*

1. *The functor $\Gamma\colon \mathcal{C} \to \mathrm{Hex}(\mathcal{C})$ preserves the terminal object and sends homotopy pullbacks to pullbacks; moreover, the image of $\Gamma$ is isomorphic to $\mathrm{Ho}(\mathcal{C})$;*

2. *The projective objects of $\mathrm{Hex}(\mathcal{C})$ are precisely the objects of the essential image of $\Gamma$;*

3. *The category $\mathrm{Hex}(\mathcal{C})$ has enough projectives.*

*Moreover, if $\mathcal{C}$ is a finitely complete category together with the path categorical structure of Example 1.5, then this statement reduces to a statement equivalent to the statement of Proposition 1.16.*

We conclude the section with a last remark. Actually, the notion of *exact completion of finitely complete categories* also works for weakly finitely complete categories, that is, by applying the procedure described in Section 1.1 to a weakly finitely complete category $\mathcal{C}$, we get an exact category $\mathrm{Ex}(\mathcal{C})$. Moreover, the following result, whose proof makes use of the usual arguments, holds (see [6]):

**Proposition 1.25.** *If $\mathcal{E}$ is an exact category with enough projectives and $\mathcal{P}$ is its full subcategory spanned by its projective objects and $\mathcal{P}$ is weakly finitely complete, then there is an equivalence of categories $\mathcal{E} \simeq \mathrm{Ex}(\mathcal{P})$.*

The following holds as a consequence:

**Corollary 1.26.** *Let $\mathcal{C}$ be a path category. Then there is an equivalence of categories $\mathrm{Hex}(\mathcal{C}) \simeq \mathrm{Ex}(\mathrm{Ho}(\mathcal{C}))$.*

*Proof.* The essential image of $\Gamma$ is the full subcategory of projective objects of $\mathrm{Hex}(\mathcal{C})$ and $\mathrm{Hex}(\mathcal{C})$ has enough projectives (Corollary 1.24). By Remark 1.19 or Corollary 1.24, the essential image of $\Gamma$ is equivalent to $\mathrm{Ho}(\mathcal{C})$, which is weakly finitely complete by *A*.22. Hence we are done by Proposition 1.25. Q.E.D.

# 2 Homotopy Natural Numbers in a Path Category

In general, a path category does not admit a weak factorisation system, since the factorisation of an arrow as a weak equivalence followed by a fibration is just unique up to the notion of homotopy of the given path category. As a consequence (see [11]) path categories model a homotopy type theory in which the elimination rule of the identity types does not hold judgementally but propositionally.

Hence, it makes sense, for a path category, to redefine up to homotopy the usual universal constructions. In other words, a given structure that, in a usual category, satisfies a universal property, corresponds to a structure, in a path category, satisfying the same universal property, but only up to (fibred) homotopy. For instance in this brief chapter we consider the path category-theoretic counterpart of the usual notion of *natural numbers object*. The last section of Chapter 3 talks about it.

**Definition 2.1.** Let $\mathcal{C}$ be a path category. Let us consider a triple $(\mathbb{N}, z, S)$, where $\mathbb{N}$ is an object of $\mathcal{C}$, $z$ is an arrow $1 \to \mathbb{N}$ and $S$ is an arrow $\mathbb{N} \to \mathbb{N}$. Moreover, let us assume that for every triple $(X, x, f)$ where $X$ is an object of $\mathcal{C}$, $x$ is an arrow $1 \to X$ and $f$ is an arrow $X \to X$, the following property holds: *whenever $p$ is a fibration $X \to \mathbb{N}$ preserving their structure, i.e. making the diagram:*

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\ & X \arrow[u, "p"'] \arrow[r, "f"] & X \arrow[u, "p"'] \end{tikzcd}$$

*commute, there is a section $\mathbb{N} \xrightarrow{a} X$ of $p$ such that $az \simeq x$ and $fa \simeq aS$.* Then we say that $(\mathbb{N}, z, S)$ is a *homotopy natural numbers object* of $\mathcal{C}$.

The following strengthening of the previous definition was contemplated for the first time in [12].

**Definition 2.2.** Let $\mathcal{C}$ be a path category. Let us consider a triple $(\mathbb{N}, z, S)$, where $\mathbb{N}$ is an object of $\mathcal{C}$, $z$ is an arrow $1 \to \mathbb{N}$ and $S$ is an arrow $\mathbb{N} \to \mathbb{N}$. Moreover, let us assume that for every triple $(X, x, f)$ where $X$ is an object of $\mathcal{C}$, $x$ is an arrow $1 \to X$ and $f$ is an arrow $X \to X$, the following property holds: *whenever $p$ is a fibration $X \to \mathbb{N}$ preserving their structure, i.e. making the diagram:*

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\ & X \arrow[u, "p"'] \arrow[r, "f"] & X \arrow[u, "p"'] \end{tikzcd}$$

*commute, there is a section $\mathbb{N} \xrightarrow{a} X$ of $p$ such that $az \simeq_{\mathbb{N}} x$ and $fa \simeq_{\mathbb{N}} aS$.* Then we say that $(\mathbb{N}, z, S)$ is a *strong homotopy natural numbers object* of $\mathcal{C}$.

By Proposition A.18 every strong homotopy natural numbers object is a homotopy natural numbers object. Moreover, observe that a more familiar definition of the notion of homotopy natural numbers object is available:

**Proposition 2.3.** *Let $\mathcal{C}$ be a path category and let $(\mathbb{N}, z, S)$ be a triple as in Definition 2.1. Then $(\mathbb{N}, z, S)$ is a homotopy natural numbers object if and only if, for every triple $(X, x, f)$,* *there is unique up to homotopy a map* $\mathbb{N} \xrightarrow{h} X$ *such that* $hz \simeq x$ *and* $fh \simeq hS$*, i.e. making the following diagram:*

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "h"] & \mathbb{N} \arrow[d, "h"] \\
& X \arrow[r, "f"] & X
\end{tikzcd}$$

*commute up to homotopy.*

*Proof.*

*Only if.* Let $(X, x, f)$ be a triple as in the statement. With respect to the diagrams:

$$\begin{tikzcd}
X & X \times \mathbb{N} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{N} \\
X \arrow[u, "f"] & X \times \mathbb{N} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{N} \arrow[u, "S"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X & X \times \mathbb{N} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{N} \\
& 1 \arrow[ul, "x"] \arrow[ur, "z"'] &
\end{tikzcd}$$

let us consider the unique arrows $X \times \mathbb{N} \xrightarrow{f \times S} X \times \mathbb{N}$ and $1 \xrightarrow{\langle x,z \rangle} X \times \mathbb{N}$ making them commute respectively. In particular, the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "\langle x,z \rangle"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\
& X \times \mathbb{N} \arrow[u, "\pi_2"'] \arrow[r, "f \times S"] & X \times \mathbb{N} \arrow[u, "\pi_2"']
\end{tikzcd}$$

commutes and $\pi_2$ is a fibration, hence there is a section $a$ of $\pi_2$ such that $az \simeq_{\mathbb{N}} \langle x, z \rangle$ and $(f \times S)a \simeq_{\mathbb{N}} aS$. By Lemma A.18 it is the case that $az \simeq \langle x, z \rangle$ and $(f \times S)a \simeq aS$, hence $(\pi_1 a)z \simeq \pi_1 \langle x, z \rangle = x$ and $f(\pi_1 a) = \pi_1(f \times S)a \simeq (\pi_1 a)S$. Therefore $(\mathbb{N} \xrightarrow{h} X) := \pi_1 a$ does the job.

If $\mathbb{N} \xrightarrow{h'} X$ does the job as well, then observe that $\langle h, h' \rangle (Sz) \simeq (f \times f)\langle h, h' \rangle z \simeq (f \times f)\langle x, x \rangle = \langle s, t \rangle rfx$ and $\langle h, h' \rangle z \simeq \langle x, x \rangle = \langle s, t \rangle rx$. By Theorem A.11 there are $1 \xrightarrow{a} PX$ and $1 \xrightarrow{b} PX$ such that $\langle h, h' \rangle (Sz) = \langle s, t \rangle a$ and $\langle h, h' \rangle z = \langle s, t \rangle b$. If the following diagram:

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "p"'] & PX \arrow[d, "\langle s,t \rangle"] \\
\mathbb{N} \arrow[r, "\langle h,h' \rangle"] & X \times X
\end{tikzcd}$$

is a pullback, then the arrows $\circ := (1 \xrightarrow{\langle Sz, a \rangle} Q)$ and $\bullet := (1 \xrightarrow{\langle z, b \rangle} Q)$ define over $Q$ a structure, so that $p$ is a fibration from this to $(\mathbb{N}, z, S)$. Hence $p$ has a section and $h \simeq h'$.

*If.* Let us assume that the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\
& X \arrow[u, "p"'] \arrow[r, "f"] & X \arrow[u, "p"']
\end{tikzcd}$$

commutes and that $p$ is a fibration. Then there is, unique up to homotopy, an arrow $\mathbb{N} \xrightarrow{h} X$ such that the following:

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "h"] & \mathbb{N} \arrow[d, "h"] \\ & X \arrow[r, "f"] & X \end{tikzcd}$$

commutes up to homotopy. As the following diagrams:

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "z"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "ph"] & \mathbb{N} \arrow[d, "ph"] \\ & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "z"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "1_{\mathbb{N}}"] & \mathbb{N} \arrow[d, "1_{\mathbb{N}}"] \\ & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \end{tikzcd}$$

commute up to homotopy, it is the case that $ph \simeq 1_{\mathbb{N}}$. By Theorem A.11 and being $p$ a fibration, there is $\mathbb{N} \xrightarrow{a} X$ such that $a \simeq h$ and $pa = 1_{\mathbb{N}}$. Moreover, the following:

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "a"] & \mathbb{N} \arrow[d, "a"] \\ & X \arrow[r, "f"] & X \end{tikzcd}$$

commutes up to homotopy and we are done. Q.E.D.

The remaining part of the section is devoted to other characterisations and criteria for the notion of (strong) homotopy natural numbers object. We need all of them in order to prove the results of last section of Chapter 3.

**Proposition 2.4.** *Let $(\mathbb{N}, 0, S)$ be a homotopy natural numbers object of a given path category $\mathcal{C}$ and let $(X, x, f)$ be an ordinary triple. Then $(X, x, f)$ is a homotopy natural numbers object if and only if there is a weak equivalence $\mathbb{N} \xrightarrow{w} X$ making the following diagram:*

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "w"] & \mathbb{N} \arrow[d, "w"] \\ & X \arrow[r, "f"] & X \end{tikzcd}$$

*commute up to homotopy.*

*Proof.*

*If.* Let $(Y, y, g)$ be an ordinary triple. Then there is, unique up to homotopy, an arrow $\mathbb{N} \xrightarrow{v} Y$ such that (♣):

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "y"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "v"] & \mathbb{N} \arrow[d, "v"] \\ & Y \arrow[r, "g"] & Y \end{tikzcd}$$

commutes up to homotopy. Hence the following diagram (♠):

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[dr, "y"'] & X \arrow[r, "f"] \arrow[d, "vw^{-1}"] & X \arrow[d, "vw^{-1}"] \\ & Y \arrow[r, "g"] & Y \end{tikzcd}$$

commutes up to homotopy, being $w^{-1}$ a pseudo inverse of $w$. Moreover, if $h$ is another arrow $X \to Y$ making the diagram ♠ commute up to homotopy, then $hw$ is an arrow $\mathbb{N} \to Y$ making the diagram ♣ commute up to homotopy. Therefore it is the case that $v \simeq hw$ and hence $h \simeq hww^{-1} \simeq vw^{-1}$.

*Only if.* As usual, one applies the homotopy universal property twice and gets that the unique arrow from $(\mathbb{N}, 0, S)$ to $(X, x, f)$ preserving the structure is actually a weak equivalence. Q.E.D.

**Proposition 2.5.** *Let $\mathcal{C}$ be a path category and let $(\mathbb{N}, z, S)$ be a homotopy natural numbers object. Moreover, let $z'$ and $S'$ be parallel arrows to $z$ and $S'$ respectively such that $z \simeq z'$ and $S \simeq S'$. Then $(\mathbb{N}, z', S')$ is a homotopy natural numbers object.*

*Proof.* Let us consider an object $X$ of $\mathcal{C}$ together with an arrow $1 \xrightarrow{x} X$ and an arrow $X \xrightarrow{f} X$. Being $(\mathbb{N}, z, S)$ a homotopy natural numbers object, there is unique up to homotopy a map $\mathbb{N} \xrightarrow{h} X$ such that $hz \simeq x$ and $hS \simeq fh$. Hence it is the case that $hz' \simeq hz \simeq x$ and $hS' \simeq hS \simeq fh$. Moreover, if $h'$ is an arrow $\mathbb{N} \to X$ such that $h'z' \simeq x$ and $h'S' \simeq fh'$, then it is the case that $h'z \simeq h'z' \simeq x$ and $h'S \simeq h'S' \simeq fh'$. Hence $h' \simeq h$. Q.E.D.

**Proposition 2.6.** *Let $\mathcal{C}$ be a path category and let $(\mathbb{N}, z, S)$ be a strong homotopy natural numbers object. Moreover, let $z'$ and $S'$ be parallel arrows to $z$ and $S'$ respectively such that $z \simeq z'$ and $S \simeq S'$. Then $(\mathbb{N}, z', S')$ is a strong homotopy natural numbers object.*

*Proof.* Let $h$ and $h'$ be homotopies $z' \simeq z$ and $S' \simeq S$ and let us assume that there is a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r, "z'"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S'"] & \mathbb{N} \\
 & X \arrow[u, "p"'] \arrow[r, "f"] & X \arrow[u, "p"']
\end{tikzcd}$$

for a given fibration $p$. Then the diagram:

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "x"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\
 & X \arrow[u, "p"'] \arrow[r, "f"] & X \arrow[u, "p"']
\end{tikzcd}$$

commutes up to homotopy. In particular, $px = z' \overset{h}{\simeq} z$ and $pf = S'p \overset{h'p}{\simeq} Sp$. Hence the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "\Gamma_p\langle x,h\rangle"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\
 & X \arrow[u, "p"'] \arrow[r, "\Gamma_p\langle f,h'p\rangle"] & X \arrow[u, "p"']
\end{tikzcd}$$

commutes by Theorem A.11. Let $a$ be a section of $p$ such that $l\colon az \simeq_{\mathbb{N}} \Gamma_p\langle x, h\rangle$ and $l'\colon aS \simeq_{\mathbb{N}} \Gamma_p\langle f, h'p\rangle a = \Gamma_p\langle fa, h'\rangle$. Now, let $\sigma$ be a filler $PX \to PX$ of the square:

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, "r"'] & PX \arrow[d, "\langle s,t\rangle"] \\
PX \arrow[r, "\langle t,s\rangle"] & X \times X
\end{tikzcd}$$

whose existence is ensured by Theorem A.12. Then we observe that:

$$az' = ash = at\sigma h \simeq_{\mathbb{N}} \Gamma_p\langle as, 1_{P\mathbb{N}}\rangle\sigma h = \Gamma_p\langle ath, \sigma h\rangle = \Gamma_p\langle az, \sigma h\rangle$$

by Lemma A.20. Moreover, with respect to the pullbacks:

$$\begin{tikzcd} P_{\mathbb{N}}X \times_{\mathbb{N}} P\mathbb{N} \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & P\mathbb{N} \arrow[d, "s"] \\ P_{\mathbb{N}}X \arrow[r, "o"] & \mathbb{N} \end{tikzcd} \qquad \begin{tikzcd} P_p \arrow[r, "p_2"] \arrow[d, "p_1"'] & P\mathbb{N} \arrow[d, "s"] \\ X \arrow[r, "p"] & \mathbb{N} \end{tikzcd}$$

by Theorem A.21 it is the case that $\Gamma_p\langle s_{\mathbb{N}}\beta_1, \beta_2\rangle \simeq_{\mathbb{N}} \Gamma_p\langle t_{\mathbb{N}}\beta_1, \beta_2\rangle$. We observe that $ol = ps_{\mathbb{N}}l = paz = z = th = s\sigma h$. Hence, with respect to the left pullback we can consider the arrow $\langle l, \sigma h\rangle$. Finally, we observe that:

$$\begin{aligned} \Gamma_p\langle az, \sigma h\rangle = \Gamma_p\langle s_{\mathbb{N}}\beta_1, \beta_2\rangle\langle l, \sigma h\rangle \simeq_{\mathbb{N}} \Gamma_p\langle t_{\mathbb{N}}\beta_1, \beta_2\rangle\langle l, \sigma h\rangle &= \Gamma_p\langle \Gamma_p\langle x, h\rangle, \sigma h\rangle \\ &= \Gamma_p\langle \Gamma_p\langle p_1, p_2\rangle, \sigma p_2\rangle\langle x, h\rangle \\ &\simeq_{\mathbb{N}} p_1\langle x, h\rangle = x \end{aligned}$$

again by Theorem A.21, hence we conclude that $az' \simeq_{\mathbb{N}} x$. Following the same argument with $h'$ and $l'$ instead of $h$ and $l$, one gets that $aS' \simeq_{\mathbb{N}} fa$. We conclude that $(\mathbb{N}, z', S')$ is a strong homotopy natural numbers object. Q.E.D.

**Proposition 2.7.** *Let $\mathcal{C}$ be a path category and let $(\mathbb{N}, z, S)$ be a strong homotopy natural numbers object. Moreover, let $\mathbb{N} \xrightarrow{u} X$ be a section of an acyclic fibration $X \xrightarrow{l} \mathbb{N}$. Then $(X, uz, uSl)$ is a strong homotopy natural numbers object as well.*

*Proof.* Let us consider a commutative diagram:

$$\begin{tikzcd} 1 \arrow[r, "uz"] \arrow[dr, "y"'] & X \arrow[r, "uSl"] & X \\ & Y \arrow[u, "p"'] \arrow[r, "f"] & Y \arrow[u, "p"'] \end{tikzcd}$$

in $\mathcal{C}$, where $p$ is a fibration, and let us consider the following pullback:

$$\begin{tikzcd} \mathbb{N} \times_X Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{\mathbb{N}}"'] & Y \arrow[d, "p"] \\ \mathbb{N} \arrow[r, "u"] & X \end{tikzcd}$$

which exists because $p$ is a fibration. If $P_X Y \to P$ is a fibreb path object of $Y$ w.r.t. $p$, then we can consider $u^*(P_X Y) \to \mathbb{N}$ as a fibred path object of $\mathbb{N} \times_X Y$ w.r.t. $\pi_{\mathbb{N}}$ (see Example 3.2 and Proposition 3.3).

Now, we observe that $uz = py$ and that $uS\pi_{\mathbb{N}} = uSlu\pi_{\mathbb{N}} = uSlp\pi_Y = pf\pi_Y$, hence the arrows $1 \xrightarrow{\langle z, y\rangle} \mathbb{N} \times_X Y$ and $\mathbb{N} \times_X Y \xrightarrow{\langle S\pi_{\mathbb{N}}, f\pi_Y\rangle}$ exist (w.r.t. the previous pullback) and the following diagram:

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "\langle z, y\rangle"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\ & \mathbb{N} \times_X Y \arrow[u, "\pi_{\mathbb{N}}"'] \arrow[r, "\langle S\pi_{\mathbb{N}}, f\pi_Y\rangle"] & \mathbb{N} \times_X Y \arrow[u, "\pi_{\mathbb{N}}"'] \end{tikzcd}$$

commutes. Hence, being $(\mathbb{N}, z, S)$ a strong homotopy natural numbers object and being $\pi_{\mathbb{N}}$ a fibration, there is a section $a$ of $\pi_{\mathbb{N}}$ such that $az \simeq_{\mathbb{N}\times_X Y} \langle z, y\rangle$ and $aS \simeq_{\mathbb{N}\times_X Y} \langle S\pi_{\mathbb{N}}, f\pi_Y\rangle a$. As a consequence of the choice of fibred path objects, it is the case that $\pi_Y az \simeq_X y$ and $\pi_Y aS \simeq_X f\pi_Y a$.

We can assume that the homotopy $h\colon ul \simeq 1_X$ is such that $hu \simeq_{X\times X} ru$ by Lemma A.23. Let us consider the pullback:

$$\begin{array}{ccc} P_p & \xrightarrow{p_2} & PX \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{p} & X \end{array}$$

and let us observe that $p\pi_Y al = u\pi_{\mathbb{N}} al = ul = sh$, hence the arrow $X \xrightarrow{\langle \pi_Y al, h\rangle} P_p$ exists. Let $P_p \xrightarrow{\Gamma} Y$ be a transport structure of $p$ and let us observe that $p\Gamma\langle \pi_Y al, h\rangle = p_p\langle \pi_Y al, h\rangle = tp_2\langle \pi_Y al, h\rangle = th = 1_X$ (see Theorem A.9), that is, the arrow $\Gamma\langle \pi_Y al, h\rangle$ is a section of the fibration $p$. Then we are done if we prove that $\Gamma\langle \pi_Y al, h\rangle uz \simeq_Y y$ and $\Gamma\langle \pi_Y al, h\rangle uSl \simeq f\Gamma\langle \pi_Y al, h\rangle$. In fact:

- $\Gamma\langle \pi_Y al, h\rangle uz = \Gamma\langle \pi_Y az, huz\rangle \simeq_X \Gamma\langle y, ruz\rangle = \Gamma\langle y, rpy\rangle = \Gamma\langle 1_Y, rp\rangle y \simeq_X y$, where the last fibred homotopy holds by Theorem A.9.

- By Lemma A.19 and being $u$ a weak equivalence, it is enough to prove that:

  $$\Gamma\langle \pi_Y al, h\rangle uSlu \simeq_X f\Gamma\langle \pi_Y al, h\rangle u.$$

  In fact it is the case that $\Gamma\langle \pi_Y al, h\rangle uSlu = \Gamma\langle \pi_Y aS, huS\rangle \simeq_X \Gamma\langle \pi_Y aS, ruS\rangle = \Gamma\langle \pi_Y aS, ru\pi_{\mathbb{N}} aS\rangle = \Gamma\langle \pi_Y aS, rp\pi_Y aS\rangle = \Gamma\langle 1_Y, rp\rangle \pi_Y aS \simeq_X \pi_Y aS \simeq_X f\pi_Y a \simeq_X f\Gamma\langle 1_Y, rp\rangle \pi_Y a = f\Gamma\langle \pi_Y a, rp\pi_Y a\rangle = f\Gamma\langle \pi_Y a, ru\pi_{\mathbb{N}} a\rangle = f\Gamma\langle \pi_Y a, ru\rangle \simeq_X f\Gamma\langle \pi_Y a, hu\rangle = f\Gamma\langle \pi_Y alu, hu\rangle = f\Gamma\langle \pi_Y al, h\rangle u$. Q.E.D.

# 3 Generalised Gluing for Path Categories

As mentioned in the Introduction, the present chapter is mostly about the notions of *Grothendieck fibration* and *generalised gluing*. We extend some of the results contained in [12] and involving the notion of *gluing* to the corresponding generalised notion. This idea was proposed by Benno van den Berg.

At first we prove some basic results involving a path category-theoretic strengthening of the notion of Grothendieck fibration, called *fibred path category*. This is precisely what we need in order to present and discuss the notion of generalised gluing for path categories. We also need the following:

**Definition 3.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be path categories and let $F$ be a functor $\mathcal{C} \to \mathcal{D}$. We say that $F$ is *exact* if it preserves the terminal object, the fibrations, the weak equivalences and the pullbacks of fibrations.

**Example 3.2.** Let $\mathcal{C}$ be a path category and let $g$ be an arrow $A \to B$ of $\mathcal{C}$. By Proposition A.6, the restricted pullback functor $\mathcal{C}(B) \xrightarrow{g^*} \mathcal{C}(A)$ is an exact functor.

**Proposition 3.3.** *Exact functors preserve (fibred) path objects and (fibred) homotopies.*

*Proof.* Let $\mathcal{C}$ and $\mathcal{D}$ be path categories and let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$. Let $X \xrightarrow{\alpha} A$ be an object of $\mathcal{C}$ and let $(P_A X \to A, r_A, \langle s_A, t_A \rangle)$ be a path object on $X \xrightarrow{\alpha} A$ in $\mathcal{C}(A)$. As $r_A$ and $\langle s_A, t_A \rangle$ are a weak equivalence and a fibrations respectively and commute with $X \xrightarrow{\alpha} A$ and $P_A X \to A$, it is the case that $Fr_A$ and $F\langle s_A, t_A \rangle$ are again a weak equivalence and a fibration respectively and commute with $FX \xrightarrow{F\alpha} FA$ and $FP_A X \to FA$. As $F$ preserves pullbacks of fibrations, it is the case that $F(X \times_A X) = FX \times_{FA} FX$ and, if $p_1$ and $p_2$ are the projections of $X \times_A X$, then $Fp_1$ and $Fp_2$ are the projection of $F(X \times_A X)$. Since $(Fp_1)(F\langle s_A, t_A \rangle) = F(p_1 \langle s_A, t_A \rangle) = Fs_A$ and analogously $(Fp_2)(F\langle s_A, t_A \rangle) = Ft_A$ and moreover $(F\alpha)(Ft_A) = (F\alpha)(Fs_A)$ (as $\alpha t_A = \alpha s_A$), it is the case that $F\langle s_A, t_A \rangle = \langle Fs_A, Ft_A \rangle$ with respect to the pullback:

$$\begin{tikzcd}
F(X \times_A X) \arrow[r, "p_2"] \arrow[d, "p_1"'] & FX \arrow[d, "\alpha"] \\
FX \arrow[r, "\alpha"] & FA.
\end{tikzcd}$$

Finally, since:

$$\begin{aligned}
(\langle Fs_A, Ft_A \rangle)(Fr_A) &= \langle (Fs_A)(Fr_A), (Ft_A)(Fr_A) \rangle \\
&= \langle F(s_A r_A), F(t_A r_A) \rangle = \langle F(1_X), F(1_X) \rangle \\
&= \langle 1_{FX}, 1_{FX} \rangle = \delta_{FX}
\end{aligned}$$

we conclude that $(FP_A X \to FA, Fr_A, \langle Fs_A, Ft_A \rangle)$ is a path object on $F\alpha$ in $\mathcal{C}(FA)$.

Now, let us assume that $f, g$ are parallel arrows $Y \to X$ such that $\alpha f = \alpha g$ and $f \simeq_A g$ and let $h$ be fibred homotopy $Y \to P_A X$. Then $FY \xrightarrow{Fh} FP_A X$ is a fibred homotopy $Ff \simeq_{FA} Fg$. Q.E.D.

## 3.1 Grothendieck Fibrations and Fibred Path Categories

We start this section by recollecting the notions of *cartesian morphism* and *Grothendieck fibration* and some remarks about them.

**Definition 3.4.** Let $\mathcal{D}$ and $\mathcal{E}$ be categories and let $Q$ be a functor $\mathcal{E} \to \mathcal{D}$. Let $X \xrightarrow{u} Y$ be an arrow of $\mathcal{E}$ and let us assume that, for every arrow $Z \xrightarrow{v} Y$ of $\mathcal{E}$ and every arrow $QZ \xrightarrow{h} QX$ of $\mathcal{D}$ such that the following diagram:

$$\begin{tikzcd} QZ \arrow[rr, "Qv"] \arrow[dr, "h"'] & & QY \\ & QX \arrow[ur, "Qu"'] & \end{tikzcd}$$

commutes, there is unique an arrow $Z \xrightarrow{h'} X$ of $\mathcal{E}$ such that the following diagram:

$$\begin{tikzcd} Z \arrow[rr, "v"] \arrow[dr, "h'"'] & & Y \\ & X \arrow[ur, "u"'] & \end{tikzcd}$$

commutes and $Q(Z \xrightarrow{h'} X) = h$. Then we say that the arrow $X \xrightarrow{u} Y$ is $Q$-*cartesian*. Moreover we say that $Q$ is a *Grothendieck fibration* if, for every object $Y$ of $\mathcal{E}$ and every arrow $D \xrightarrow{f} QY$ of $\mathcal{D}$, there is a $Q$-cartesian arrow $f^+Y \xrightarrow{f^+} Y$ such that $Q(f^+Y \xrightarrow{f^+} Y) = f$.

**Remark 3.5.** From now on, given a Grothendieck fibration $Q\colon \mathcal{E} \to \mathcal{D}$, we assume that there is already a choice, for every object $Y$ of $\mathcal{E}$ and every arrow $D \xrightarrow{f} QY$ of $\mathcal{D}$, of a $Q$-cartesian arrow $f^+Y \xrightarrow{f^+} Y$ such that $Q(f^+Y \xrightarrow{f^+} Y) = f$. In other words, we assume that there is a functional relation sending every couple $(Y, f)$, where $Y$ is an object of $\mathcal{E}$ and $f$ is an arrow of $\mathcal{D}$ of target $QY$, to a $Q$-cartesian arrow $f^+Y \xrightarrow{f^+} Y$ such that $Q(f^+Y \xrightarrow{f^+} Y) = f$. Such a choice is called *cleavage*. In particular, observe that, for every object $Y$ of $\mathcal{E}$, the arrow $Y \xrightarrow{1_Y} Y$ is a $Q$-cartesian arrow such that $Q(Y \xrightarrow{1_Y} Y) = 1_{QY}$. Hence we assume that our cleavage is such that $1^+_{QY} = 1_Y$ for every $Y$ in $\mathcal{E}$.

Under this assumption, let $X_1$ and $X_2$ be objects of $\mathcal{E}$ and let $I_1 \xrightarrow{v_1} QX_1$ and $I_2 \xrightarrow{v_2} QX_2$ be arrows of $\mathcal{D}$. Then, for every arrow $X_1 \xrightarrow{f} X_2$ of $\mathcal{E}$ and every arrow $I_1 \xrightarrow{u} I_2$ of $\mathcal{D}$ such that the following diagram:

$$\begin{tikzcd} I_1 \arrow[r, "v_1"] \arrow[d, "u"'] & QX_1 \arrow[d, "Qf"] \\ I_2 \arrow[r, "v_2"] & QX_2 \end{tikzcd}$$

commutes, there is unique an arrow $v_1^+X_1 \xrightarrow{(v_1,v_2)^+f} v_2^+X_2$ such that the following diagram:

$$\begin{tikzcd} v_1^+X_1 \arrow[r, "v_1^+"] \arrow[d, "(v_1,v_2)^+f"'] & X_1 \arrow[d, "f"] \\ v_2^+X_2 \arrow[r, "v_2^+"] & X_2 \end{tikzcd}$$

commutes and $Q(v_1^+ X_1 \xrightarrow{(v_1,v_2)^+ f} v_2^+ X_2) = u$.

**Remark 3.6.** Let $Q\colon \mathcal{E} \to \mathcal{D}$ be a Grothendieck fibration (with a cleavage) and assume that there is a choice, for every object $X$ of $\mathcal{C}$, of an arrow $I_X \xrightarrow{v_X} QX$ of $\mathcal{D}$. By Remark 3.5 there is a functor $F$ sending every object $X$ of $\mathcal{C}$ to $v_X^+ X$ and every arrow $(X \xrightarrow{f} Y, I_X \xrightarrow{u} I_Y)$ to $v_X^+ X \xrightarrow{(v_X,v_Y)^+ f} v_Y^+ Y$, with the property that $QF(f,u) = u$.

In particular, there is a functor $F\colon \mathcal{D} \to \mathcal{D}$ sending every object $X$ of $\mathcal{C}$ to $(1_X)^+ X$ and every arrow $X \xrightarrow{f} Y$ to $(1_X)^+ X \xrightarrow{(1_X,1_Y)^+ f} (1_Y)^+ Y$, with the property that $QF = Q$.

The following notion is due to Taichi Uemura.

**Definition 3.7.** Let $\mathcal{E}$ be a category with a terminal object. Let us assume that there are in $\mathcal{E}$ a class of of arrows called *fibrations* and a class of arrows called *weak equivalences*. Let $\mathcal{D}$ be a path category and let $Q\colon \mathcal{E} \to \mathcal{D}$ be a Grothendieck fibration preserving the terminal object, the fibrations and the weak equivalences. Moreover let us assume that the following properties are satisfied:

1. The composition of two fibrations of $\mathcal{E}$ is a fibration as well.
2. Every pullback of a fibration of $\mathcal{E}$ exists and is a fibration as well and $Q$ preserves it.
3. Every pullback of an *acyclic fibration* (that is, an arrow being both a fibration and a weak equivalence) of $\mathcal{E}$ is an acyclic fibration as well.
4. For every choice of arrows $f$, $g$ and $h$ of $\mathcal{E}$, if the compositions $gf$ and $hg$ exist and are weak equivalences, then $f$, $g$, $h$ and $hgf$ are weak equivalences as well.
5. *Section lifting property.* Every isomorphism of $\mathcal{E}$ is an acyclic fibration and, for every acyclic fibration $X \xrightarrow{f} Y$ of $\mathcal{E}$ and every section $QY \xrightarrow{s} QX$ of $QX \xrightarrow{Qf} QY$ in $\mathcal{D}$, there is a section $Y \xrightarrow{s'} X$ of $X \xrightarrow{f} Y$ such that $Q(Y \xrightarrow{s'} X) = s$. In particular, every acyclic fibration of $\mathcal{E}$ has a section.
6. *Path lifting property.* For every fibration $X \to A$ of $\mathcal{E}$ and every path object $(P(QX \to QA) = P_{QA}(QX) \to QA, r_{QA}, \langle s_{QA}, t_{QA}\rangle)$ in $\mathcal{D}(QA)$ of $QX \to QA$, there is a *path object* $(P(X \to A) = P_A X \to A, r_A, \langle s_A, t_A\rangle)$ of $X \to A$ (that is, $X \xrightarrow{r_A} P_A X$ is a weak equivalence and $P_A X \xrightarrow{\langle s_A,t_A\rangle} X \times_A X$ is a fibration and $(X \xrightarrow{r_A} P_A X \xrightarrow{\langle s_A,t_A\rangle} X \times_A X) = (X \xrightarrow{\langle 1_X,1_X\rangle} X \times_A X)$ holds) whose image through $Q$ is $(P_{QA}(QX) \to QA, r_{QA}, \langle s_{QA}, t_{QA}\rangle)$.
7. Every arrow of $\mathcal{E}$ of target a terminal object is a fibration.
8. Under the hypothesis of Remark 3.5, if $f$ and $u$ are fibrations (weak equivalences), then $(v_1, v_2)^+ f$ is a fibration (weak equivalence) as well.

In particular $\mathcal{E}$ is a path category. We say that $Q$ is a *fibred path category over* $\mathcal{D}$.

The remaining part of the section contains the main features enjoyed by a fibred path category $Q$:

- The functor of Remark 3.5 associated to $Q$ preserves both the fibrations and the weak equivalences.

- Every fibration in the domain of $Q$ (weak equivalence) factors as a fibration (weak equivalence) whose image is an identity, followed by a $Q$-cartesian fibration (weak equivalence).
- Acyclic fibrations are weakly cartesian.
- (Fibred) homotopies are reflected by $Q$.
- (Strong) homotopy natural numbers objects are preserved by $Q$.

**Remark 3.8.** Let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. Under the hypothesis of Remark 3.5, if $f = 1_{X_1}$ and $v_2 = 1_{QX_1}$, then the following diagrams:

$$\begin{tikzcd} I_1 \arrow[r, "v_1"] \arrow[d, "u"'] & QX_1 \arrow[d, "Q(1_{X_1})"] \\ QX_1 \arrow[r, "1_{QX_1}"] & QX_1 \end{tikzcd} \qquad \begin{tikzcd} v_1^+ X_1 \arrow[r, "v_1^+"] \arrow[d, "(v_1,v_2)^+ f"'] & X_1 \arrow[d, "1_{X_1}"] \\ X_1 \arrow[r, "1_{X_1}"] & X_1 \end{tikzcd}$$

commute. Hence, if $v_1$ is a fibration (weak equivalence) then $u = v_1$ is a fibration (weak equivalence). Then $(v_1, v_2)^+ f$ is a fibration (weak equivalence) by Definition 3.7. Therefore it is the case that $v_1^+ = (v_1, v_2)^+ f$ is a fibration (weak equivalence). This proves that, whenever $Y$ is an object of $\mathcal{E}$ and $D \xrightarrow{f} QY$ is a fibration (weak equivalence) of $\mathcal{D}$, then $f^+$ is a fibration (weak equivalence) of $\mathcal{E}$.

**Remark 3.9.** Let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. Under the hypothesis of Remark 3.5, if $v_1 = u = 1_{QX_1}$, then the following diagrams:

$$\begin{tikzcd} QX_1 \arrow[r, "1_{QX_1}"] \arrow[d, "1_{QX_1}"'] & QX_1 \arrow[d, "Qf"] \\ QX_1 \arrow[r, "v_2"] & QX_2 \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "1_{X_1}"] \arrow[d, "(v_1,v_2)^+ f"'] & X_1 \arrow[d, "f"] \\ v_2^+ X_2 \arrow[r, "v_2^+"] & X_2 \end{tikzcd}$$

commute. Observe that $Q(v_2^+) = v_2 = Qf$ and that $Q((v_1, v_2)^+ f) = 1_{QX_1}$. Hence every arrow $f$ is the composition $f = f_2 f_1$ of two arrows $f_1$ and $f_2$ such that $Qf_1$ is an identity, $Qf_2 = Qf$ and $f_2$ is cartesian.

If $f$ is a fibration (weak equivalence) then $Qf$ is a fibration (weak equivalence) by Definition 3.7. Hence $v_2 = Qf$ is a fibration (weak equivalence) and $v_2^+$ is a fibration (weak equivalence) by Remark 3.8. Moreover $(v_1, v_2)^+ f$ is a fibration (weak equivalenece) by Definition 3.7. Hence every fibration (weak equivalence) $f$ is the composition $f = f_2 f_1$ of two fibrations (weak equivalences) $f_1$ and $f_2$ such that $Qf_1$ is an identity, $Qf_2 = Qf$ and $f_2$ is cartesian.

**Proposition 3.10** (*Acyclic fibrations are weakly cartesian*)**.** *Let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$ and let $f$ be an acyclic fibration $X \to Y$ of $\mathcal{E}$. Then for every arrow $Z \xrightarrow{g} Y$ and every commutative diagram:*

$$\begin{tikzcd} QZ \arrow[rr, "Qg"] \arrow[dr, "h"'] & & QY \\ & QX \arrow[ur, "Qf"'] & \end{tikzcd}$$

*in $\mathcal{D}$, there is an arrow $Z \xrightarrow{h'} X$ such that $Qh' = h$ and the following diagram:*

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "h'"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

*commutes.*

*Proof.* By Remark 3.9 it is the case that $f$ is the composition $f = (X \xrightarrow{f_1} X' \xrightarrow{f_2} Y)$ of two acyclic fibrations $f_1$ and $f_2$ such that $Qf_1 = 1_{QX}$, $Qf_2 = Qf$ and $f_2$ is cartesian. Suppose that a diagram as in statament commutes. Hence, as the following diagram:

$$\begin{tikzcd} QZ \arrow[rr, "Qg"] \arrow[dr, "h"'] & & QY \\ & QX = QX' \arrow[ur, "Qf_2=Qf"'] & \end{tikzcd}$$

commutes as well and $f_2$ is cartesian, there is unique an arrow $Z \xrightarrow{h''} X'$ such that:

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "h''"'] & & Y \\ & X' \arrow[ur, "f_2"'] & \end{tikzcd}$$

commutes and $Qh'' = h$. Let $X' \xrightarrow{s} X$ be a section of $f_1$. Since $Qs$ needs to be a section of $Qf_1 = 1_{QX}$, it is the case that $Qs = 1_{QX}$. Finally, observe that the diagram:

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "sh''"'] & & Y \\ & X \arrow[ur, "f=f_2f_1"'] & \end{tikzcd}$$

commutes and $Q(sh'') = (Qs)(Qh'') = Qh'' = h$. We are done. Q.E.D.

**Corollary 3.11** (*Homotopy lifting property*)**.** *Let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$, let $Y \xrightarrow{f} X$ be an arrow of $\mathcal{E}$ and let $\alpha$ be a fibration $X \to A$. Moreover, let $h$ be a homotopy $Qf \simeq_{QA} b$, being $b$ an arrow $QY \to QX$. Then there are arrows $h'$ and $g$ of $\mathcal{E}$ such that $Qh' = h$, $Qg = b$ and $h'\colon f \simeq_A g$.*

*Proof.* Let $(P_AX \to A, r_A, \langle s_A, t_A\rangle)$ be a fibred path object w.r.t. the fibration $\alpha$. By Proposition 3.3 and since fibred homotopy does not depend on the choice of the path object, without loss of generality it is the case that $h$ is an arrow $QY \to QP_AX$ and the following diagram:

$$\begin{tikzcd} QY \arrow[r, "\langle Qf,b\rangle"] \arrow[d, "h"'] & QX \times_{QA} QX \\ QP_AX \arrow[ur, "\langle Qs_A,Qt_A\rangle"'] & \end{tikzcd}$$

commutes. In particular, the following triangle:

$$\begin{tikzcd} QY \arrow[rr, "Qf"] \arrow[dr, "h"'] & & QX \\ & QP_AX \arrow[ur, "Qs_A"'] & \end{tikzcd}$$

commutes and, by Proposition 3.10 and being $s_A$ an acyclic fibration, there is $Y \xrightarrow{h'} P_AX$ such that $Qh' = h$ and the following triangle:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "h'"'] & & X \\ & P_AX \arrow[ur, "s_A"'] & \end{tikzcd}$$

commutes. Let $g := (Y \xrightarrow{h'} P_AX \xrightarrow{t_A} X)$. Then $\alpha f = \alpha s_A h' = \alpha t_A h' = \alpha g$ and hence $h' \colon f \simeq_A g$. Moreover $Qg = (Qt_A)(Qh') = (Qt_A)h = b$. Q.E.D.

**Proposition 3.12.** *Let $Q \colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$ and let $(\mathbb{N}, z, S)$ be a (strong) homotopy natural numbers object of $\mathcal{E}$. Then $(Q\mathbb{N}, Qz, QS)$ is a (strong) homotopy natural numbers object of $\mathcal{D}$.*

*Proof.* Let $x$ and $f$ be arrows $1 \to X$ and $X \to X$ of $\mathcal{D}$ and let us assume the following diagram:

$$\begin{tikzcd} Q1 \arrow[r, equal] & 1 \arrow[r, "Qz"] \arrow[dr, "x"'] & Q\mathbb{N} \arrow[r, "QS"] & Q\mathbb{N} \\ & & X \arrow[u, "p"'] \arrow[r, "f"] & X \arrow[u, "p"'] \end{tikzcd}$$

for some fibration $p$. By Remark 3.5, there is unique an arrow $p^+\mathbb{N} \xrightarrow{p^+S} p^+\mathbb{N}$ such that the square:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\ p^+\mathbb{N} \arrow[u, "p^+"'] \arrow[r, "p^+S"] & p^+\mathbb{N} \arrow[u, "p^+"'] \end{tikzcd}$$

commutes and $Q(p^+S) = f$. By Remark 3.8, it is the case that $p^+$ is a fibration and, as it is cartesian, there is unique an arrow $1 \xrightarrow{y} p^+\mathbb{N}$ such that the following:

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "y"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\ & p^+\mathbb{N} \arrow[u, "p^+"'] \arrow[r, "p^+S"] & p^+\mathbb{N} \arrow[u, "p^+"'] \end{tikzcd}$$

commutes and $Qy = x$. Being $(\mathbb{N}, z, S)$ a natural numbers object of $\mathcal{E}$, there is a section $a$ of $p^+$ that makes this diagram:

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "y"'] & \mathbb{N} \arrow[r, "S"] \arrow[d, "a"] & \mathbb{N} \arrow[d, "a"] \\ & p^+\mathbb{N} \arrow[r, "p^+S"] & p^+\mathbb{N} \end{tikzcd}$$

commutes up to (fibred - if we are considering a strong homotopy natural numbers object) homotopy w.r.t. $\mathbb{N}$. Hence $Fa$ is a section of $p = Qp^+$ and, by Proposition 3.3, the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "Qz"] \arrow[dr, "x"'] & Q\mathbb{N} \arrow[r, "QS"] \arrow[d, "Fa"] & Q\mathbb{N} \arrow[d, "Fa"] \\
& X \arrow[r, "f"] & X
\end{tikzcd}$$

commutes up to (fibred - if we are considering a strong homotopy natural numbers object) homotopy w.r.t. $Q\mathbb{N}$ and we are done. Q.E.D.

## 3.2 Generalised Gluing and its Notion of Homotopy

We can finally define the notion of generalised gluing for path categories. As mentioned in the introduction, this generalises the notion of gluing construction presented and studied in [12], where $Q$ is always assumed to be the identity functor. The gluing construction arises in several areas of category theory, that is, for several classes of categories. The general idea motivating it contemplation is that it may help to prove canonicity results for the deductive system modeled by the corresponding class of categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be path categories and let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$. Moreover, let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. We define the category $\mathrm{GL}(F, Q)$ as follows:

- An object of $\mathrm{GL}(F, Q)$ is a triple $(X, A, \alpha)$ where $X$ is an object of $\mathcal{E}$, $A$ is an object of $\mathcal{C}$ and $\alpha$ is a fibration $QX \to FA$ of $\mathcal{D}$.
- If $(X, A, \alpha)$ and $(Y, B, \beta)$ are objects of $\mathrm{GL}(F, Q)$, an arrow $(X, A, \alpha) \to (Y, B, \beta)$ of $\mathrm{GL}(F, Q)$ is a couple $(f_0, f_1)$ where $f_0$ is an arrow $X \to Y$ of $\mathcal{E}$ and $f_1$ is an arrow $A \to B$ of $\mathcal{C}$ such that the diagram:

$$\begin{tikzcd}
QX \arrow[r, "\alpha"] \arrow[d, "Qf_0"'] & FA \arrow[d, "Ff_1"] \\
QY \arrow[r, "\beta"] & FB
\end{tikzcd}$$

  commutes in $\mathcal{D}$.

The composition of two arrows of $\mathrm{GL}(F, Q)$ is the couple of the compositions of their components.

We say that an arrow $(X, A, \alpha) \xrightarrow{(f_0, f_1)} (Y, B, \beta)$ of $\mathrm{GL}(F, g)$ is a weak equivalence if $f_0$ is a weak equivalence of $\mathcal{E}$ and $f_1$ is a weak equivalence of $\mathcal{C}$. Moreover we say that it is a fibration if $f_0$ is a fibration of $\mathcal{E}$, $f_1$ is a fibration of $\mathcal{C}$ and, whenever the following square:

$$\begin{tikzcd}
QY \times_{FB} FA \arrow[r] \arrow[d] & FA \arrow[d, "Ff_1"] \\
QY \arrow[r, "\beta"] & FB
\end{tikzcd}$$

is a pullback (it exists because $Ff_1$ is a fibration, as $F$ preserves fibrations), the arrow $QX \xrightarrow{\langle Qf_0, \alpha \rangle} QY \times_{FB} FA$ is a fibration of $\mathcal{D}$. Then the following holds:

**Theorem 3.13.** *Whenever $F$ is an exact functor $\mathcal{C} \to \mathcal{D}$ between path categories and $Q\colon \mathcal{E} \to \mathcal{D}$ is a fibred path category over $\mathcal{D}$, then* $\mathrm{GL}(F,Q)$ *is a path category.*

*Proof.* Observe that $(1,1,1_1)$ is a terminal object of $\mathrm{GL}(F,Q)$, being the first component a terminal object of $\mathcal{E}$, the second one a terminal object of $\mathcal{C}$ and the third one the identity arrow $Q1 = 1 \to 1 = F1$ in $\mathcal{E}$ (of course $1_1$ is a fibration by Definition 1.3). Indeed, whenever $(X,A,\alpha)$ is an object of $\mathrm{GL}(F,Q)$, the couple $(!,!)$ is the unique arrow $(X,A,\alpha) \to (1,1,1_1)$, being the first component the unique arrow $X \to 1$ in $\mathcal{E}$ and being the second one the unique arrow $A \to 1$ in $\mathcal{C}$. Let us verify the seven properties of Definition 1.3.

1. Let $(X,A,\alpha) \xrightarrow{(f_0,f_1)} (Y,B,\beta)$ and $(Y,B,\beta) \xrightarrow{(g_0,g_1)} (Z,C,\gamma)$ be fibrations of $\mathrm{GL}(F,Q)$. Clearly $g_0f_0$ and $g_1f_1$ are fibration as $f_0$, $g_0$, $f_1$ and $g_1$ are fibrations.

   Let us consider the following diagram:

$$\begin{tikzcd}
QY \times_{FB} FA \arrow[r] \arrow[d] & QZ \times_{FC} FB \times_{FB} FA \arrow[r] \arrow[d] & FA \arrow[d, "Ff_1"] \\
QY \arrow[r, "\langle Qg_0,\beta\rangle"] & QZ \times_{FC} FB \arrow[r] \arrow[d] & FB \arrow[d, "Fg_1"] \\
 & QZ \arrow[r, "\gamma"] & FC
\end{tikzcd}$$

   where three smallest squares are pullbacks. Hence the two biggest squares are pullback as well. Let us call $A$ the one of the couple:

$$(\beta = (QY \xrightarrow{\langle Qg_0,\beta\rangle} QZ \times_{FC} FB \to FB),\ Ff_1)$$

   and $B$ the one of the couple $(\gamma,\ F(g_1f_1))$. Now observe that the arrow:

$$x = (QX \xrightarrow{\langle Qf_0,\alpha\rangle} QY \times_{FB} FA \to QZ \times_{FC} FB \times_{FB} FA)$$

   verifies:

   (a) $(QX \xrightarrow{x} QZ \times_{FC} FB \times_{FB} FA \to QZ) = (Qg_0)(Qf_0) = Q(g_0f_0)$

   (b) $(QX \xrightarrow{x} QZ \times_{FC} FB \times_{FB} FA \to FA) = \alpha$

   and therefore $x = \langle Q(g_0f_0), \alpha\rangle$ w.r.t. the pullback $B$. But then, as $\langle Qf_0, \alpha\rangle$ is a fibration of $\mathcal{D}$ (because $(f_0,f_1)$ is a fibration of $\mathrm{GL}(F,Q)$) and the arrow:

$$QY \times_{FB} FA \to QZ \times_{FC} FB \times_{FB} FA$$

   is also a fibration of $\mathcal{D}$ (it is a pullback of the fibration $\langle Qg_0, \beta\rangle$), it is the case that $\langle Q(g_0f_0), \alpha\rangle$ is a fibration as well.

   Hence $(g_0,g_1)(f_0,f_1)$ is a fibration $(X,A,\alpha) \to (Z,C,\gamma)$.

2. Let $(X,A,\alpha) \xrightarrow{(f_0,f_1))} (Y,B,\beta)$ be an arrow of $\mathrm{GL}(F,Q)$ and let $(Z,C,\gamma) \xrightarrow{(g_0,g_1)} (Y,B,\beta)$ be a fibration of $\mathrm{GL}(F,Q)$. Then $g_0$ and $g_1$ are fibrations, hence the following pullbacks:

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "i_0"] \arrow[d, "h_0"'] & Z \arrow[d, "g_0"] \\
X \arrow[r, "f_0"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times_B C \arrow[r, "i_1"] \arrow[d, "h_1"'] & C \arrow[d, "g_1"] \\
A \arrow[r, "f_1"] & B
\end{tikzcd}$$

exist and $h_0$ and $h_1$ are fibrations. Let us consider the diagram (♠):

$$\begin{tikzcd}
QX \times_{QY} QZ \arrow[rr, dotted] \arrow[dr, "Qi_0=:j"] \arrow[dd, "Qh_0=:i"'] & & FA \times_{FB} FC \arrow[dr, "Fi_1"] \arrow[dd, dashed, "Fh_1"] & \\
& QZ \arrow[rr, "\gamma"] \arrow[dd, "Qg_0"] & & FC \arrow[dd, "Fg_1"] \\
QX \arrow[rr, dashed, "\alpha"] \arrow[dr, "Qf_0"'] & & FA \arrow[dr, dashed, "Ff_1"] & \\
& QY \arrow[rr, "\beta"'] & & FB
\end{tikzcd}$$

where the left and right squares are pullbacks (both $F$ and $Q$ preserve pullbacks). Hence, by the universal property of the right one there exist unique an arrow:

$$QX \times_{QY} QZ \xrightarrow{\alpha \times_\beta \gamma} FA \times_{FB} FC$$

such that this diagram commutes (that is, the arrow $\langle \alpha i, \gamma j \rangle$ w.r.t. the right pullback). Moreover, observe that in the following diagram (♣):

$$\begin{tikzcd}
QX \times_{QY} QZ \arrow[dr, "j"] \arrow[dd] & & & \\
& QZ \arrow[dd, "\langle Qg_0, \gamma \rangle"] & & \\
QX \times_{FB} FC \arrow[rr, dashed] \arrow[dr] \arrow[dd] & & FA \times_{FB} FC \arrow[dr] \arrow[dd, dashed] & \\
& QY \times_{FB} FC \arrow[rr] \arrow[dd] & & FC \arrow[dd, "Fg_1"] \\
QX \arrow[rr, dashed, "\alpha"] \arrow[dr, "Qf_0"'] & & FA \arrow[dr, dashed, "Ff_1"] & \\
& QY \arrow[rr, "\beta"'] & & FB
\end{tikzcd}$$

where all five smaller vertical squares are pullbacks, the arrow:

$$x := (QX \times_{QY} QZ \to QX \times_{FB} FC \to FA \times_{FB} FC)$$

is precisely $\alpha \times_\beta \gamma$. In fact:

(a) $(QX \times_{QY} QZ \xrightarrow{x} FA \times_{FB} FC \to FA) = (QX \times_{QY} QZ \to QX \times_{FB} FC \to QX \xrightarrow{\alpha} FA) = (QX \times_{QY} QZ \xrightarrow{i} QX \xrightarrow{\alpha} FA)$

(b) $(QX \times_{QY} QZ \xrightarrow{x} FA \times_{FB} FC \to FC) = (QX \times_{QY} QZ \to QX \times_{FB} FC \to QY \times_{FB} FC \to FC) = (QX \times_{QY} QZ \xrightarrow{j} QZ \xrightarrow{\langle Qg_0, \gamma \rangle} QY \times_{FB} FC \to FC) = (QX \times_{QY} QZ \xrightarrow{j} QZ \xrightarrow{\gamma} FC)$

that shows that indeed $x = \langle \alpha i, \gamma j \rangle = \alpha \times_\beta \gamma$. Now, observe that $QX \times_{QY} QZ \to QX \times_{FB} FC$ is a fibration beacuse it is a pullback of $\langle Qg_0, \gamma \rangle$, which is a fibration because $(g_0, g_1)$ is a fibration. Moreover $QX \times_{FB} FC \to FA \times_{FB} FC$ is also a fibration, because it is a pullback of the fibration $\alpha$. This proves that $\alpha \times_\beta \gamma = x$ is a fibration and therefore $(X \times_Y Z, A \times_B C, \alpha \times_\beta \gamma)$ is an object of $\mathrm{GL}(F, Q)$. From the commutativity of ♠ it follows that $(h_0, h_1)$ and $(i_0, i_1)$ are arrows of $\mathrm{GL}(F, Q)$ and it's clear that they form a pullback of the pair $((f_0, f_1), (g_0, g_1))$. Indeed, if $(l_0, l_1)$ and $(m_0, m_1)$ are arrows of same source such that $(f_0, f_1)(l_0, l_1) = (g_0, g_1)(m_0, m_1)$ then of course $\langle (l_0, l_1), (m_0, m_1) \rangle = (\langle l_0, m_0 \rangle, \langle l_1, m_1 \rangle)$.

We are left to prove that:

$$(X \times_Y Z, A \times_B C, \alpha \times_\beta \gamma) \xrightarrow{(h_0,h_1)} (X, A, \alpha)$$

is a fibration. Of course $h_0$ and $h_1$ are fibrations, as they are pullbacks of fibrations ($g_0$ and $g_1$). Moreover we know that the pullback of the couple $(\alpha, Fh_1)$ is the back square of the diagram ♣. Observe that the arrow $QX \times_{QY} QZ \to QX \times_{FB} FC$, that we already proved to be a fibration, is such that:

(a) $(QX \times_{QY} QZ \to QX \times FBFC \to QX) = Qh_0$
(b) $(QX \times_{QY} QZ \to QX \times FBFC \to F_A \times_{FB} FC) = \alpha \times_\beta \gamma$

and hence it is $\langle Qh_0, \alpha \times_\beta \gamma \rangle$, that is therefore a fibration. Hence we conclude that $(h_0, h_1)$ is a fibration.

3. If $(g_0, g_1)$ is both a fibration and a weak equivalence, then in particular $g_0$ and $g_1$ are both fibrations and weak equivalences, hence their pullbacks $h_0$ and $h_1$ are both fibrations and weak equivalences. In particular $(h_0, h_1)$ is a weak equivalence. Moreover $(h_0, h_1)$ is a fibration by 2.. Hence $(h_0, h_1)$ is both a fibration and a weak equivalence.

4. This is immediate, as a weak equivalence is a couple of weak equivalences and the composition of two couples is the couple of the compositions.

5. Let $(X, A, \alpha) \xrightarrow{(f_0,f_1)} (Y, B, \beta)$ be an isomorphism of $\mathrm{GL}(F, Q)$. Then there exists an arrow $(Y, B, \beta) \xrightarrow{(g_0,g_1)} (X, A, \alpha)$ such that $f_0 g_0$, $f_1 g_1$, $g_0 f_0$ and $g_1 f_1$ are identity arrows. Then $f_0$ and $f_1$ are isomorphism and hence acyclic fibrations. In order to conclude that $(f_0, f_1)$ is an acyclic fibration, we observe that $Qf_0 = (QX \xrightarrow{\langle Qf_0,\alpha \rangle} QY \times_{FB} FA \to QY)$ and $QY \times_{FB} FA \to QY$ is an isomorphism (it is a pullback of $Ff_1$, an isomorphism). Therefore $\langle Qf_0, \alpha \rangle = (QX \xrightarrow{Qf_0} QY \to QY \times_{FB} FA)$. Hence $\langle Qf_0, \alpha \rangle$ is a fibration because $Qf_0$ is a fibration and $QY \to QY \times_{FB} FA$ is a fibration as well (it is an isomorphism).

Now, let us assume that $(X, A, \alpha) \xrightarrow{(f_0,f_1)} (Y, B, \beta)$ is an acyclic fibration. Then the arrow $\langle Qf_0, \alpha \rangle$ is a fibration. Moreover in the following diagram:

$$\begin{tikzcd}
QX \arrow[dr, "\langle Qf_0,\alpha \rangle"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "Qf_0"'] & & \\
& QY \times_{FB} FA \arrow[r] \arrow[d] & FA \arrow[d, "Ff_1"] \\
& QY \arrow[r, "\beta"] & FB
\end{tikzcd}$$

it is the case that $Qf_0$ is a weak equivalence ($f_0$ is a weak equivalence and $Q$ preserves the weak equivalences) and $QY \times_{FB} FA \to QY$ is a weak equivalence as well (it is an acyclic fibration because it is a pullback of the acyclic fibration $Ff_1$). Hence $\langle Qf_0, \alpha\rangle$ is a weak equivalence by 4. of Definition 1.3 and then an acyclic fibration. Let $QY \times_{FB} FA \xrightarrow{s} QX$ be a section of $\langle Qf_0, \alpha\rangle$. Moreover let $B \xrightarrow{s_1} A$ be a section of $f_1$ (the latter is an acyclic fibration) and let us consider the arrow:

$$QY \xrightarrow{\langle 1_{QY}, (Fs_1)\beta\rangle} QY \times_{FB} FA$$

which exists because $\beta 1_{QY} = (Ff_1)(Fs_1)\beta$. Then $(QY \xrightarrow{s\langle 1_{QY},(Fs_1)\beta\rangle} QX \xrightarrow{Qf_0} QY) = (QY \xrightarrow{\langle 1_{QY},(Fs_1)\beta\rangle} QY \times_{FB} FA \xrightarrow{s} QX \xrightarrow{\langle Qf_0,\alpha\rangle} QY \times_{FB} FA \to QY) = (QY \xrightarrow{\langle 1_{QY},(Fs_1)\beta\rangle} QY \times_{FB} FA \to QY) = 1_{QY}$, that is, $s\langle 1_{QY}, (Fs_1)\beta\rangle$ is a section of $Qf_0$. Being $f_0$ an fibration, by 5. of Definition 3.7 there is a section $Y \xrightarrow{s_0} X$ of $f_0$ such that $Qs_0 = s\langle 1_{QY}, (Fs_1)\beta\rangle$.

Now, observe that the following diagram:

$$\begin{tikzcd}
QY \arrow[r, "\beta"] \arrow[d, "Qs_0"'] & FB \arrow[d, "Fs_1"] \\
QX \arrow[r, "\alpha"] & FA
\end{tikzcd}$$

commutes, as $\alpha(Qs_0) = (QY \xrightarrow{s\langle 1_{QY},(Fs_1)\beta\rangle} QX \xrightarrow{\alpha} FA) = (QY \xrightarrow{\langle 1_{QY},(Fs_1)\beta\rangle} QY \times_{FB} FA \xrightarrow{s} QX \xrightarrow{\langle Qf_0,\alpha\rangle} QY \times_{FB} FA \to FA) = (QY \xrightarrow{\langle 1_{QY},(Fs_1)\beta\rangle} QY \times_{FB} FA \to FA) = (Fs_1)\beta$. Hence $(s_0, s_1)$ is an arrow $(Y, B, \beta) \to (X, A, \alpha)$. Moreover, $s_0$ is a section of $f_0$ and $s_1$ is a section of $f_1$. Hence $(s_0, s_1)$ is a section of $(f_0, f_1)$.

6. Let $(X, A, \alpha)$ be an object of $\mathrm{GL}(F, Q)$ and let $(PA, r, \langle s, t\rangle)$ be a path object of $A$ in $\mathcal{C}$. Then $(FPA, Fr, \langle Fs, Ft\rangle)$ is a path object of $FA$ in $\mathcal{D}$. Hence, by Remark A.10 and being $\alpha$ a fibration $QX \to FA$, we can build a path object $(PQX, r', \langle s', t'\rangle)$ of $QX$ as follows.

   As $QX \xrightarrow{\alpha} FA$ is an object of $\mathcal{C}(FA)$, it admits a path object:

$$(P(QX \xrightarrow{\alpha} FA) = (P_{FA}(QX) \to FA),\ r_{FA},\ \langle s_{FA}, t_{FA}\rangle)$$

   in $\mathcal{C}(FA)$. In particular observe that, since the following diagram:

$$\begin{tikzcd}
P_{FA}(QX) \arrow[dr, "\langle s_{FA}, t_{FA}\rangle"] \arrow[drr, bend left, "s_{FA}"] \arrow[ddr, bend right, "t_{FA}"'] & & \\
& QX \times_{FA} QX \arrow[r] \arrow[d] & QX \arrow[d, "\alpha"] \\
& QX \arrow[r, "\alpha"'] & FA
\end{tikzcd}$$

   commutes in $\mathcal{D}$ (being $QX \times_{FA} QX$ a product in $\mathcal{C}(FA)$) it is the case that $\alpha s_{FA} =$

$\alpha t_{FA}$. Now, if the following diagrams

$$\begin{tikzcd} P_\alpha \arrow[r, "p_2"] \arrow[d, "p_1"'] & FPA \arrow[d, "Fs"] \\ QX \arrow[r, "\alpha"] & FA \end{tikzcd} \qquad \begin{tikzcd} P_\alpha \times_{QX} P_{FA}(QX) \arrow[r] \arrow[d] & P_{FA}(QX) \arrow[d, "s_{FA}"] \\ P_\alpha \arrow[r, "\Gamma"] & QX \end{tikzcd}$$

are pullbacks, $P_\alpha \xrightarrow{\Gamma} QX$ is a transport structure and:

$$(P_\alpha \xrightarrow{p_\alpha} FA) := (P_\alpha \xrightarrow{p_2} FPA \xrightarrow{Ft} FA)$$
$$(QX \xrightarrow{w_\alpha} P_\alpha) := \langle QX \xrightarrow{1_{QX}} QX, QX \xrightarrow{\alpha} FA \xrightarrow{r} FPA\rangle$$

and we define:

$$PQX := P_\alpha \times_{QX} P_{FA}(QX);$$
$$(QX \xrightarrow{r'} PQX) := \langle w_\alpha, r_X\rangle \text{ [as } \Gamma w_\alpha = 1_{QX} = s_{FA} r_{FA}\text{]};$$
$$(PQX \xrightarrow{s'} QX) := (P_\alpha \times_{QX} P_{FA}(QX) \to P_\alpha \xrightarrow{p_1} QX);$$
$$(PQX \xrightarrow{t'} QX) := (P_\alpha \times_{QX} P_{FA}(QX) \to P_{FA}(QX) \xrightarrow{t_{FA}} QX).$$

then $(PQX, r', \langle s', t'\rangle)$ is a path object of $QX$ in $\mathcal{C}$. Moreover, if:

$$(PQX \xrightarrow{P\alpha} FPA) := (P_\alpha \times_{QX} P_{FA}(QX) \to P_\alpha \xrightarrow{p_2} FPA)$$
$$(P_\alpha \xrightarrow{\nabla} PQX) := \langle 1_{P_\alpha}, (P_\alpha \xrightarrow{\Gamma} QX \xrightarrow{r_{FA}} P_{FA}(QX))\rangle \text{ [as } \Gamma 1_{P_\alpha} = s_{FA}(r_{FA}\Gamma)\text{]}$$

then the triple $(\Gamma, P_\alpha, \nabla)$ verifies the thesis of Theorem A.9. In particular the following diagram:

$$\begin{tikzcd} QX \arrow[r, "\alpha"] \arrow[d, "r'"'] & FA \arrow[d, "Fr"] \\ PQX \arrow[r, "P\alpha"] \arrow[d, "\langle s', t'\rangle"'] & FPA \arrow[d, "\langle Fs, Ft\rangle"] \\ QX \times QX \arrow[r, "\alpha\times\alpha"] & FA \times FA \end{tikzcd}$$

commutes.

By 6. of Definition 3.7, there is a path object $(PX, r, \langle s, t\rangle)$ of $X$ in $\mathcal{E}$ whose image through $Q$ is $(PQX, r', \langle s', t'\rangle)$. Hence $(PX, PA, P\alpha)$ is an object of $\mathrm{GL}(F, Q)$. Since $X \xrightarrow{r} PX$ and $A \xrightarrow{r} PA$ are weak equivalences and $PX \xrightarrow{\langle s,t\rangle} X \times X$ and $PA \xrightarrow{\langle s,t\rangle} A \times A$ are fibrations, it is the case that:

$$((PX, PA, P\alpha), (r, r), (\langle s, t\rangle, \langle s, t\rangle))$$

is a path object of $(X, A, \alpha)$ as long as, with respect to the following pullback:

$$\begin{tikzcd} (QX \times QX) \times_{FA\times FA} FPA \arrow[r] \arrow[d] & FPA \arrow[d, "\langle Fs, Ft\rangle"] \\ QX \times QX \arrow[r, "\alpha\times\alpha"] & FA \times FA \end{tikzcd}$$

the induced arrow $PQX \xrightarrow{\langle\langle s',t'\rangle,P\alpha\rangle} (QX \times QX) \times_{FA\times FA} FPA$ is a fibration.

Let us consider the commutative diagram:

$$\begin{tikzcd}
P_\alpha \times_{QX} P_{FA}(QX) \arrow[rr, dotted] \arrow[dr] \arrow[dd] & & P_\alpha \times_{FA} QX \arrow[dr, "Fi_1"] \arrow[dd, dashed, "Fh_1"] & \\
& P_{FA}(QX) \arrow[rr, "t_{FA}"] \arrow[dd, "s_{FA}"] & & QX \arrow[dd, "\alpha"] \\
P_\alpha \arrow[rr, dashed, "1_{P_\alpha}"] \arrow[dr, "\Gamma"'] & & P_\alpha \arrow[dr, dashed, "p_\alpha"'] & \\
& QX \arrow[rr, "\alpha"'] & & FA
\end{tikzcd}$$

where the left and the right squares are pullbacks. Hence there is unique an arrow $P_\alpha \times_{QX} P_{FA}(QX) \xrightarrow{1_{P_\alpha}\times_\alpha t_{FA}} P_\alpha \times_{FA} QX$ such that the diagram commutes. Moreover, in the following commutative diagram:

$$\begin{tikzcd}
P_\alpha \times_{QX} P_{FA}X \arrow[r] \arrow[d, "1_{P_\alpha}\times_\alpha t_{FA}"'] & P_{FA}(QX) \arrow[d, "\langle s_{FA}, t_{FA}\rangle"] & \\
P_\alpha \times_{FA} QX \arrow[r] \arrow[d] & QX \times_{FA} QX \arrow[r] \arrow[d] & QX \arrow[d, "\alpha"] \\
P_\alpha \arrow[r, "\Gamma"] \arrow[rr, bend right, "p_\alpha"'] & QX \arrow[r, "\alpha"] & FA
\end{tikzcd}$$

as the lower rectangle (that is, the right square of the previous diagram) and the right lower square are pullbacks, the left lower square is a pullback as well. Therefore, since the left rectangle is a pullback (it is the left square of the previous diagram) the upper square is a pullback as well. Hence $1_{P_\alpha} \times_\alpha t_{FA}$ is a fibration, because $\langle s_{FA}, t_{FA}\rangle$ is a fibration. Hence we are done, as (see [12]) there is an isomorphism $(QX \times QX) \times_{FA\times FA} FPA \to P_\alpha \times_{FA} QX$ such that the following diagram:

$$\begin{tikzcd}
PQX \arrow[r, equal] \arrow[d, "\langle\langle s',t'\rangle,P\alpha\rangle"'] & P_\alpha \times_{QX} P_{FA}(QX) \arrow[d, "1_{P_\alpha}\times_\alpha t_{FA}"] \\
(QX \times QX) \times_{FA\times FA} FPA \arrow[r] & P_\alpha \times_{FA} QX
\end{tikzcd}$$

commutes.

7. Let $(X, A, \alpha)$ be an object of $\mathrm{GL}(F, Q)$ and let us consider the unique arrow:

$$(X, A, \alpha) \to (1, 1, 1_1).$$

Then clearly $X \to 1$ and $A \to 1$ are fibrations, as they are arrows of target terminal objects in path categories. Moreover, as the following diagram:

$$\begin{tikzcd}
QX \arrow[dr, "\langle QX\to 1,\alpha\rangle"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right] & & \\
& 1\times_1 FA = FA \arrow[r, "1_{FA}"] \arrow[d] & FA \arrow[d] \\
& 1 \arrow[r, "1_1"] & 1
\end{tikzcd}$$

is commutative, it is the case that $\langle QX \to 1, \alpha\rangle = \alpha$ is a fibration. Hence $(X, A, \alpha) \to (1, 1, 1_1)$ is a fibration of $\mathrm{GL}(F, Q)$. Q.E.D.

The remaining part of the section is devoted to the characterisation of the notions of homotopy and fibred homotopy of the generalised gluing of a given exact functor and a given fibred path category over its codomain. Again, these results provide a generalisation of the ones contained in [12].

**Lemma 3.14.** *Let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$ between path categories and let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. Let $(Y, B, \beta)$ and $(X, A, \alpha)$ be objects of $\mathrm{GL}(F, Q)$, let $f_1$ and $g_1$ be parallel arrows $B \to A$ in $\mathcal{C}$ and let $\varphi_0$ and $\gamma_0$ be parallel arrows $QY \to QX$ of $\mathcal{E}$ such that the diagrams:*

$$\begin{tikzcd}
QY \arrow[r, "\beta"] \arrow[d, "\varphi_0"'] & FB \arrow[d, "Ff_1"] \\
QX \arrow[r, "\alpha"] & FA
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
QY \arrow[r, "\beta"] \arrow[d, "\gamma_0"'] & FB \arrow[d, "Fg_1"] \\
QX \arrow[r, "\alpha"] & FA
\end{tikzcd}$$

*commute.*

*Let us assume that there are homotopies $h_1\colon f_1 \simeq g_1$ and $h\colon \Gamma_\alpha\langle\varphi_0, (Fh_1)\beta\rangle \simeq_{FA} \gamma_0$. Then the arrow:*

$$QY \xrightarrow{\langle\langle\varphi_0,(Fh_1)\beta\rangle,h\rangle} QPX$$

*is a homotopy $\varphi_0 \simeq \gamma_0$.*

*Proof.* Let us consider the pullbacks:

$$\begin{tikzcd}
P_\alpha \arrow[r, "p_2"] \arrow[d, "p_1"'] & FPA \arrow[d, "Fs"] \\
QX \arrow[r, "\alpha"] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
QPX \arrow[r, equal] & P_\alpha \times_{QX} P_{FA}(QX) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_{FA}(QX) \arrow[d, "s_{FA}"] \\
& P_\alpha \arrow[r, "\Gamma_\alpha"] & QX
\end{tikzcd}$$

and, with respect to the left one, observe that $(Fs)(Fh_1)\beta = (Ff_1)\beta = \alpha\varphi_0$. Hence $\langle\varphi_0, (Fh_1)\beta\rangle$ is actually an arrow $QY \to P_\alpha$. Secondly, observe that:

$$\begin{aligned}
\alpha\Gamma_\alpha\langle\varphi_0, (Fh_1)\beta\rangle &= p_\alpha\langle\varphi_0, (Fh_1)\beta\rangle \\
&= (Ft)p_2\langle\varphi_0, (Fh_1)\beta\rangle \\
&= (Ft)(Fh_1)\beta = (Fg_1)\beta \\
&= \alpha\gamma_0
\end{aligned}$$

hence such a homotopy $h$ can exist. Moreover, with respect to the right pullback, observe that $\Gamma_\alpha\langle\varphi_0,(Fh_1)\beta\rangle = s_{FA}h$, because $h$ is a homotopy $\Gamma_\alpha\langle\varphi_0,(Fh_1)\beta\rangle \simeq_{FA} \gamma_0$. Hence the arrow:

$$QY \xrightarrow{\langle\langle\varphi_0,(Fh_1)\beta\rangle,h\rangle} QPX$$

actually exists.

Observe that:

$$\begin{aligned}(QY \xrightarrow{\langle\langle\varphi_0,(Fh_1)\beta\rangle,h\rangle} QPX \xrightarrow{Qs=s'} QX) &= (QY \xrightarrow{\langle\langle\varphi_0,(Fh_1)\beta\rangle,h\rangle} QPX \xrightarrow{\pi_1} P_\alpha \xrightarrow{p_1} QX)\\ &= (QY \xrightarrow{\langle\varphi_0,(Fh_1)\beta\rangle} P_\alpha \xrightarrow{p_1} QX)\\ &= (QY \xrightarrow{\varphi_0} QX)\end{aligned}$$

and that:

$$\begin{aligned}(QY \xrightarrow{\langle\langle\varphi_0,(Fh_1)\beta\rangle,h\rangle} QPX \xrightarrow{Qt=t'} QX) &= (QY \xrightarrow{\langle\langle\varphi_0,(Fh_1)\beta\rangle,h\rangle} QPX \xrightarrow{t_{FA}\pi_2} QX)\\ &= (QY \xrightarrow{h} QPX \xrightarrow{t_{FA}} QX)\\ &= (QY \xrightarrow{\gamma_0} QX).\end{aligned}$$

that is, the arrow $\langle\langle\varphi_0,(Fh_1)\beta\rangle,h\rangle$ is a homotopy $\varphi_0 \simeq \gamma_0$. Q.E.D.

**Theorem 3.15.** *Let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$ between path categories and let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. Let $(f_0, f_1)$ and $(g_0, g_1)$ be parallel arrows $(Y, B, \beta) \to (X, A, \alpha)$ of $\mathrm{GL}(F, Q)$. Then $(f_0, f_1) \simeq (g_0, g_1)$ if and only if there are homotopies $h_1\colon f_1 \simeq g_1$ and $h\colon \Gamma_\alpha\langle Qf_0,(Fh_1)\beta\rangle \simeq_{FA} Qg_0$ such that the arrow:*

$$QY \xrightarrow{\langle\langle Qf_0,(Fh_1)\beta\rangle,h\rangle} QPX$$

*is the image through $Q$ of a homotopy $f_0 \simeq g_0$.*

*Proof.*

*Only if.* Let us assume that $(f_0.f_1) \simeq (g_0, g_1)$. Then there is $(h_0, h_1)\colon (Y, B, \beta) \to (PX, PA, P\alpha)$ such that:

$$((Y,B,\beta) \xrightarrow{(h_0,h_1)} (PX,PA,P\alpha) \xrightarrow{(s,s)} (X,A,\alpha)) = ((Y,B,\beta) \xrightarrow{(f_0,f_1)} (X,A,\alpha))$$
$$((Y,B,\beta) \xrightarrow{(h_0,h_1)} (PX,PA,P\alpha) \xrightarrow{(t,t)} (X,A,\alpha)) = ((Y,B,\beta) \xrightarrow{(g_0,g_1)} (X,A,\alpha))$$

and hence it is the case that:

$$(Y \xrightarrow{h_0} PX \xrightarrow{\langle s,t\rangle} X \times X) = \langle f_0, g_0\rangle$$
$$(B \xrightarrow{h_1} PA \xrightarrow{\langle s,t\rangle} A \times A) = \langle f_1, g_1\rangle$$

that is, $h_0\colon f_0 \simeq g_0$ and $h_1\colon f_1 \simeq g_1$.

Let us consider the usual pullbacks:

$$\begin{tikzcd} P_\alpha \arrow[r, "p_2"] \arrow[d, "p_1"'] & FPA \arrow[d, "Fs"] \\ QX \arrow[r, "\alpha"] & FA \end{tikzcd} \qquad \begin{tikzcd} QPX = P_\alpha \times_{QX} P_{FA}(QX) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_{FA}(QX) \arrow[d, "s_{FA}"] \\ P_\alpha \arrow[r, "\Gamma"] & QX \end{tikzcd}$$

and, with respect to the left one, let us consider the arrow $QY \xrightarrow{\langle Qf_0,(Fh_1)\beta\rangle} P_\alpha$, as it is the case that $(Fs)(Fh_1)\beta = (Ff_1)\beta = \alpha(Qf_0)$. Observe that:

$$\begin{aligned}(QY \xrightarrow{\Gamma\langle Qf_0,(Fh_1)\beta\rangle} QX \xrightarrow{\alpha} FA) &= (QY \xrightarrow{\langle Qf_0,(Fh_1)\beta\rangle} P_\alpha \xrightarrow{p_\alpha} FA)\\ &= (QY \xrightarrow{\langle Qf_0,(Fh_1)\beta\rangle} P_\alpha \to FPA \xrightarrow{Ft} FA)\\ &= (QY \xrightarrow{(Fh_1)\beta} FPA \xrightarrow{Ft} FA)\\ &= (QY \xrightarrow{\beta} FB \xrightarrow{Fg_1} FA)\\ &= (QY \xrightarrow{Qg_0} QX \xrightarrow{\alpha} FA).\end{aligned}$$

Moreover:

$$\begin{aligned}(QY \xrightarrow{Qh_0} QPX \xrightarrow{\pi_2} P_{FA}(QX) \xrightarrow{s_{FA}} QX) &= (QY \xrightarrow{Qh_0} QPX \xrightarrow{\pi_1} P_\alpha \xrightarrow{\Gamma} QX)\\ &= (QY \xrightarrow{\Gamma\langle Qf_0,(Fh_1)\beta\rangle} QX)\end{aligned}$$

as $p_1(\pi_1(Qh_0)) = s'(Qh_0) = (Qs)(Qh_0) = Qf_0$ and $p_2(\pi_1(Qh_0)) = (P\alpha)(Qh_0) = (Fh_1)\beta$, and:

$$\begin{aligned}(QY \xrightarrow{Qh_0} QPX \xrightarrow{\pi_2} P_{FA}(QX) \xrightarrow{t_{FA}} QX) &= (QY \xrightarrow{Qh_0} QPX \xrightarrow{t'=Qt} QX)\\ &= (QY \xrightarrow{Qg_0} QX).\end{aligned}$$

Hence it is the case that $\pi_2(Qh_0)\colon \Gamma\langle Qf_0,(Fh_1)\beta\rangle \simeq_{FA} Qg_0$. Moreover, observe that the arrow $\langle\langle Qf_0,(Fh_1)\beta\rangle, \pi_2(Qh_0)\rangle$ actually exists (w.r.t. the right pullback), as $\Gamma\langle Qf_0,(Fh_1)\beta\rangle = s_{FA}\pi_2(Qh_0)$, and that:

$$\pi_2\langle\langle Qf_0,(Fh_1)\beta\rangle, \pi_2(Qh_0)\rangle = \pi_2(Qh_0)$$

and that:

$$\begin{aligned}\pi_1\langle\langle Qf_0,(Fh_1)\beta\rangle, \pi_2(Qh_0)\rangle &= \langle Qf_0,(Fh_1)\beta\rangle\\ &= \pi_1(Qh_0)\end{aligned}$$

because $p_1\pi_1(Qh_0) = s'(Qh_0) = (Qs)(Qh_0) = Qf_0$ and $p_2\pi_1(Qh_0) = (P\alpha)(Qh_0) = (Fh_1)\beta$. Hence it is the case that $\langle\langle Qf_0,(Fh_1)\beta\rangle, \pi_2(Qh_0)\rangle = Qh_0$ is the image through $Q$ of a homotopy $f_0 \simeq g_0$.

*If.* Viceversa, let us assume that there are homotopies $h\colon \Gamma\langle Qf_0,(Fh_1)\beta\rangle \simeq_{FA} Qg_0$ and $h_1\colon f_1 \simeq g_1$ and that the arrow:

$$QY \xrightarrow{\langle\langle Qf_0,(Fh_1)\beta\rangle,h\rangle} QPX$$

which actually exists since $\Gamma\langle Qf_0,(Fh_1)\beta\rangle = s_{FA}h$, equals $Qh_0$ for some homotopy $h_0\colon f_0 \simeq g_0$. By Lemma 3.14, it is the case that $\langle\langle Qf_0,(Fh_1)\beta\rangle,h\rangle$ is a homotopy $Qf_0 \simeq Qg_0$. Moreover, it is the case that:

$$\begin{aligned}(P\alpha)(Qh_0) &= (P\alpha)\langle\langle Qf_0,(Fh_1)\beta\rangle,h\rangle\\ &= p_2\pi_1\langle\langle Qf_0,(Fh_1)\beta\rangle,h\rangle\\ &= p_2\langle Qf_0,(Fh_1)\beta\rangle\\ &= (Fh_1)\beta\end{aligned}$$

hence $(h_0,h_1)$ is an arrow $(Y,B,\beta) \to (PX,PA,P\alpha)$. Q.E.D.

**Remark 3.16.** Let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$ between path categories and let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. Let $(X, A, \alpha) \xrightarrow{(p_0,p_1)} (Z, C, \gamma)$ be a fibration of $\mathrm{GL}(F, Q)$.

We consider a fibred path object $(P_C A \to C, r_C, \langle s_C, t_C\rangle)$ of $A$ over $C$ through the fibration $p_1$. Hence, being the pullback along a given map an exact functor, with respect to the diagram:

$$\begin{tikzcd}
 & QZ \times_{FC} FA \arrow[rr, "q_2"] \arrow[dl, "q_1"'] \arrow[dd, "r_{QZ}"] & & FA \arrow[dl, dashed, "Fp_1"] \arrow[dd, "Fr_C"] \\
QZ \arrow[rr, dashed, "\gamma"] & & FC & \\
 & QZ \times_{FC} FP_C A \arrow[rr, "b_2"'] \arrow[ul, "b_1"] \arrow[dd, shift right, "s_{QZ}"'] \arrow[dd, shift left, "t_{QZ}"] & & FP_C A \arrow[ul, dashed, "\bullet"] \arrow[dd, shift right, "Fs_C"'] \arrow[dd, shift left, "Ft_C"] \\
 & & & \\
 & QZ \times_{FC} FA \arrow[rr, "q_2"'] \arrow[uuul, bend left, "q_1"] & & FA \arrow[uuul, dashed, bend left, "Fp_1"]
\end{tikzcd}$$

whose left-hand side is the pullback through $\gamma$ of the right-hand side, it is the case that $(QZ \times_{FC} FP_C A, r_{QZ}, \langle s_{QZ}, t_{QZ}\rangle)$ is a path object of $QZ \times_{FC} FA \to QZ$ in $\mathcal{D}(QZ)$.

As the following diagram:

$$\begin{tikzcd}
QX \arrow[rr, "\langle Qp_0, \alpha\rangle"] \arrow[dr, "Qp_0"'] & & QZ \times_{FC} FA = \gamma^*(FA) \arrow[dl, "q_1"] \\
 & QZ &
\end{tikzcd}$$

commutes and all the arrows are fibrations, it is the case that $\langle Qp_0, \alpha\rangle$ is an arrow of $\mathcal{D}(QZ)$ and hence we can consider a transport structure $P_{\langle Qp_0,\alpha\rangle} \xrightarrow{\Gamma_{\langle Qp_0,\alpha\rangle}} QX$ of $\langle Qp_0, \alpha\rangle$ in $\mathcal{D}(QZ)$. Moreover, a path object:

$$\Big(P_{\gamma^*(FA)}QX \to \gamma^*(FA),\, r_{\gamma^*(FA)},\, \langle s_{\gamma^*(FA)}, t_{\gamma^*(FA)}\rangle\Big)$$

of $\langle Qp_0, \alpha\rangle$ in $\mathcal{D}(\gamma^*(FA))$ provides a fibred path object of the object $\langle Qp_0, \alpha\rangle$ of $(\mathcal{D}(QZ))(q_1)$. Therefore, we can apply Theorem A.9 and Remark A.10 in order to get a fibred path object of $QX$ over $QZ$, that is, a path object of $QX \xrightarrow{Qp_0} QZ$ in $\mathcal{D}(QZ)$. Indeed, if the following square:

$$\begin{tikzcd}
P_{\langle Qp_0,\alpha\rangle} \times_{QX} P_{\gamma^*(FA)}QX \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_{\gamma^*(FA)}QX \arrow[d, "s_{\gamma^*(FA)}"] \\
P_{\langle Qp_0,\alpha\rangle} \arrow[r, "\Gamma_{\langle Qp_0,\alpha\rangle}"] & QX
\end{tikzcd}$$

is a pullback in $\mathcal{D}(QZ)$, then:

$$\begin{aligned}
&QP_Z X = P_{QZ}QX = P_{\langle Qp_0,\alpha\rangle} \times_{QX} P_{\gamma^*(FA)}QX \\
&(QX \xrightarrow{Qr_Z} P_{QZ}QX) = \langle w_{\langle Qp_0,\alpha\rangle}, r_{\gamma^*(FA)}\rangle \text{ [as } \Gamma_{\langle Qp_0,\alpha\rangle} w_{\langle Qp_0,\alpha\rangle} = 1_{QX} = s_{\gamma^*(FA)} r_{\gamma^*(FA)}] \\
&(P_{QZ}QX \xrightarrow{Qs_Z} QX) = (P_{QZ}QX \xrightarrow{\pi_1} P_{\langle Qp_0,\alpha\rangle} \xrightarrow{p_1} QX) \\
&(P_{QZ}QX \xrightarrow{Qt_Z} QX) = (P_{QZ}QX \xrightarrow{\pi_2} P_{\gamma^*(FA)}QX \xrightarrow{t_{\gamma^*(FA)}} QX)
\end{aligned}$$

being the following diagram:

$$\begin{tikzcd} P_{\langle Qp_0,\alpha\rangle} \arrow[r, "p_2"] \arrow[d, "p_1"'] & QZ \times_{FC} FP_C A \arrow[d, "s_{QZ}"] \\ QX \arrow[r, "\langle Qp_0,\alpha\rangle"] & \gamma^*(FA) \end{tikzcd}$$

a pullback in $\mathcal{D}(QZ)$, and for some $(P_Z X \to Z, r_Z, \langle s_Z, t_Z\rangle)$ path object of $X \xrightarrow{p_0} Z$ in $\mathcal{E}(Z)$ (here we used that $Q$ is a fibred path category).

Moreover the fibration:

$$P_Z\langle Qp_0, \alpha\rangle = (QP_Z X \xrightarrow{\pi_1} P_{\langle Qp_0,\alpha\rangle} \xrightarrow{p_2} QZ \times_{FC} FP_C A)$$

between the path objects (fibred over $QZ$) of $QX$ and $\gamma^*(FA)$ respectively, commutes with their structure. Hence the fibration:

$$P_\gamma \alpha := (QP_Z X \xrightarrow{P_Z\langle Qp_0,\alpha\rangle} QZ \times_{FC} FP_C A \xrightarrow{b_2} FP_C A)$$

between the the fibred path objects $QX$ over $QZ$ and $FA$ over $FC$ respectively, again commutes with their structure. We gave an explicit description of a fibred path object:

$$(P_Z X, P_C A, P_\gamma \alpha)$$

of $(X, A, \alpha)$ over $(Z, C, \gamma)$ in $\mathrm{GL}(F, Q)$, i.e. a path object of $(p_0, p_1)$ in $(\mathrm{GL}(F, Q))(Z, C, \gamma)$.

**Lemma 3.17.** *Let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$ between path categories and let $Q \colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. Let $(Y, B, \beta)$ and $(X, A, \alpha)$ be objects of $\mathrm{GL}(F, Q)$, let $f_1$ and $g_1$ be parallel arrows $B \to A$ in $\mathcal{C}$ and let $\varphi_0$ and $\gamma_0$ be parallel arrows $QY \to QX$ of $\mathcal{E}$ such that the diagrams:*

$$\begin{tikzcd} QY \arrow[r, "\beta"] \arrow[d, "\varphi_0"'] & FB \arrow[d, "Ff_1"] \\ QX \arrow[r, "\alpha"] & FA \end{tikzcd} \qquad \begin{tikzcd} QY \arrow[r, "\beta"] \arrow[d, "\gamma_0"'] & FB \arrow[d, "Fg_1"] \\ QX \arrow[r, "\alpha"] & FA \end{tikzcd}$$

*commute. Moreover let us assume that there $(Qp_0)\varphi_0 = (Qp_0)\gamma_0$ and that $p_1 f_1 = p_1 g_1$, where $(p_0, p_1)$ is a fibration $(X, A, \alpha) \to (Z, C, \gamma)$.*

*Let us assume that there are homotopies:*

$$h_1 \colon f_1 \simeq_C g_1 \text{ and } h \colon \Gamma_{\langle Qp_0,\alpha\rangle}\langle \varphi_0, \langle (Qp_0)\varphi_0, (Fh_1)\beta\rangle\rangle \simeq_{\gamma^*(FA)} \gamma_0.$$

*Then the arrow:*

$$QY \xrightarrow{\langle\langle \varphi_0, \langle (Qp_0)\varphi_0, (Fh_1)\beta\rangle\rangle, h\rangle} QP_Z X$$

*is a homotopy $\varphi_0 \simeq_{QZ} \gamma_0$.*

*Proof.* Let us consider the following pullbacks:

$$\begin{tikzcd} QZ \times_{FC} FP_C A \arrow[r, "b_2"] \arrow[d, "b_1"'] & FP_C A \arrow[d, "\bullet"] \\ QZ \arrow[r, "\gamma"] & FC \end{tikzcd} \quad \begin{tikzcd} P_{\langle Qp_0,\alpha\rangle} \arrow[r, "p_2"] \arrow[d, "p_1"'] & QZ \times_{FC} FP_C A \arrow[d, "s_{QZ}"] \\ QX \arrow[r, "\langle Qp_0,\alpha\rangle"] & \gamma^*(FA) \end{tikzcd} \quad \begin{tikzcd} QP_Z X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_{\gamma^*(FA)} QX \arrow[d, "s_{\gamma^*(FA)}"] \\ P_{\langle Qp_0,\alpha\rangle} \arrow[r, "\Gamma_{\langle Qp_0,\alpha\rangle}"] & QX \end{tikzcd}$$

and, with respect to the first one, observe that $\gamma(Qp_0)\varphi_0 = \bullet(Fh_1)\beta$, hence the arrow $\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle$ actually exists. Moreover, with respect to the second one, observe that:

$$\langle\varphi_0,\alpha\rangle\varphi_0 = s_{QZ}\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle$$

hence the arrow $\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle$ exists as well.

Finally, let us observe that the arrow $\langle\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle,h\rangle$ exists (w.r.t. the third pullback), since:

$$s_{\gamma^*(FA)}h = \Gamma_{\langle Qp_0,\alpha\rangle}\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle,$$

being $h$ a homotopy. Moreover we observe that:

$$\begin{aligned}(Qs_Z)\langle\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle,h\rangle &= p_1\pi_1\langle\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle,h\rangle\\ &= p_1\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle = \varphi_0\end{aligned}$$

and that:

$$\begin{aligned}(Qt_Z)\langle\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle,h\rangle &= t_{\gamma^*(FA)}\pi_2\langle\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle,h\rangle\\ &= t_{\gamma^*(FA)}h = \gamma_0\end{aligned}$$

hence $\langle\langle\varphi_0,\langle(Qp_0)\varphi_0,(Fh_1)\beta\rangle\rangle,h\rangle$ is a homotopy $\varphi_0 \simeq_{QZ} \gamma_0$. Q.E.D.

**Theorem 3.18.** *Let $F$ be an exact functor $\mathcal{C}\to\mathcal{D}$ between path categories and let $Q\colon\mathcal{E}\to\mathcal{D}$ be a fibred path category over $\mathcal{D}$. Let $(f_0,f_1)$ and $(g_0,g_1)$ be arrows $(Y,B,\beta)\to(X,A,\alpha)$ such that $(p_0,p_1)(f_0,f_1)=(p_0,p_1)(g_0,g_1)$, where $(p_0,p_1)$ is a fibration $(X,A,\alpha)\to(Z,C,\gamma)$. Then $(f_0,f_1)\simeq_{(Z,C,\gamma)}(g_0,g_1)$ if and only if there are homotopies:*

$$h_1\colon f_1\simeq_C g_1 \text{ and } h\colon \Gamma_{\langle Qp_0,\alpha\rangle}\langle Qf_0,\langle(Qp_0)(Qf_0),(Fh_1)\beta\rangle\rangle \simeq_{\gamma^*(FA)} Qg_0$$

*such that the arrow:*

$$QY \xrightarrow{\langle\langle Qf_0,\langle(Qp_0)(Qf_0),(Fh_1)\beta\rangle\rangle,h\rangle} QP_ZX$$

*is the image through $Q$ of a homotopy $f_0\simeq_Z g_0$.*

*Proof.*

*Only if.* Let us assume that $(f_0,f_1)\simeq_{(Z,C,\gamma)}(g_0,g_1)$. Then there is $(h_0,h_1)\colon(Y,B,\beta)\to(P_ZX,P_CA,P_\gamma\alpha)$ such that:

$$((Y,B,\beta)\xrightarrow{(h_0,h_1)}(P_ZX,P_CA,P_\gamma\alpha)\xrightarrow{(s_Z,s_C)}(X,A,\alpha)) = ((Y,B,\beta)\xrightarrow{(f_0,f_1)}(X,A,\alpha))$$
$$((Y,B,\beta)\xrightarrow{(h_0,h_1)}(P_ZX,P_CA,P_\gamma\alpha)\xrightarrow{(t_Z,t_C)}(X,A,\alpha)) = ((Y,B,\beta)\xrightarrow{(g_0,g_1)}(X,A,\alpha))$$

being $(P_ZX,P_CA,P_\gamma\alpha)$ a fibred path object of $(X,A,\alpha)$ over $(Z,C,\gamma)$ as built in Remark 3.16. Hence it is the case that:

$$(Y\xrightarrow{h_0}P_ZX\xrightarrow{\langle s_Z,t_Z\rangle}X\times X) = \langle f_0,g_0\rangle$$
$$(B\xrightarrow{h_1}P_CA\xrightarrow{\langle s_C,t_C\rangle}A\times A) = \langle f_1,g_1\rangle$$

that is, $h_0\colon f_0\simeq_Z g_0$ and $h_1\colon f_1\simeq_C g_1$.

Let us consider the following pullbacks:

$$\begin{tikzcd} QZ \times_{FC} FP_C A \arrow[r, "b_2"] \arrow[d, "b_1"'] & FP_C A \arrow[d, "\bullet"] \\ QZ \arrow[r, "\gamma"] & FC \end{tikzcd} \quad \begin{tikzcd} P_{\langle Qp_0, \alpha \rangle} \arrow[r, "p_2"] \arrow[d, "p_1"'] & QZ \times_{FC} FP_C A \arrow[d, "s_{QZ}"] \\ QX \arrow[r, "\langle Qp_0, \alpha \rangle"] & \gamma^*(FA) \end{tikzcd} \quad \begin{tikzcd} QP_Z X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_{\gamma^*(FA)} QX \arrow[d, "s_{\gamma^*(FA)}"] \\ P_{\langle Qp_0, \alpha \rangle} \arrow[r, "\Gamma_{\langle Qp_0, \alpha \rangle}"] & QX \end{tikzcd}$$

and, with respect to the first one, let us observe that $\gamma(Qp_0)(Qf_0) = \bullet(Fh_1)\beta$. Indeed:

$$\begin{aligned} \bullet(Fh_1)\beta &= \bullet(P_\gamma \alpha)(Qh_0) \\ &= \gamma b_1 p_2 \pi_1 (Qh_0) \\ &= \gamma(Qp_0) p_1 \pi_1 (Qh_0) \\ &= \gamma(Qp_0)(Qs_Z)(Qh_0) \\ &= \gamma(Qp_0)(Qf_0) \end{aligned}$$

where the third equality holds because the following diagram:

$$\begin{tikzcd} P_{\langle Qp_0, \alpha \rangle} \arrow[r, "p_2"] \arrow[d, "p_1"'] & QZ \times_{FC} FP_C A \arrow[d, "s_{QZ}"'] \arrow[dd, bend left=60, "b_1"] \\ QX \arrow[r, "\langle Qp_0, \alpha \rangle"] \arrow[dr, "Qp_0"'] & \gamma^*(FA) \arrow[d, "q_1"'] \\ & QZ \end{tikzcd}$$

commutes. Therefore we can consider the arrow $QY \xrightarrow{\langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle} QZ \times_{FC} FP_C A$.

Now, with respect to the second pullback, observe that:

$$\langle Qp_0, \alpha \rangle (Qf_0) = s_{QZ} \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle$$

as it is the case that $q_1 s_{QZ} \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle = b_1 \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle = (Qp_0)(Qf_0)$ and that $q_2 s_{QZ} \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle = (Fs_C) b_2 \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle = (Fs_C)(Fh_1)\beta = (Ff_1)\beta = \alpha(Qf_0)$. Hence the arrow $QY \xrightarrow{\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle \rangle} P_{\langle Qp_0, \alpha \rangle}$ exists.

Let us consider the arrow:

$$QY \xrightarrow{Qh_0} P_{\langle Qp_0, \alpha \rangle} \times_{QX} P_{\gamma^*(FA)} QX \xrightarrow{\pi_2} P_{\gamma^*(FA)} QX$$

and let us observe that the equality:

$$s_{\gamma^*(FA)} \pi_2 (Qh_0) \overset{\clubsuit}{=} \Gamma_{\langle Qp_0, \alpha \rangle} \langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle \rangle$$

holds. Indeed, as it is the case that $s_{\gamma^*(FA)} \pi_2 = \Gamma_{\langle Qp_0, \alpha \rangle} \pi_1$, it is enough to see that $\pi_1(Qh_0) = \langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle \rangle$, that is: $p_1 \pi_1 (Qh_0) = Qf_0$ and $p_2 \pi_1 (Qh_0) = \langle (Qp_0)(Qf_0), (Fh_1)\beta \rangle$. The former is true because $p_1 \pi_1 = Qs_z$. The latter holds if and only if: $b_1 p_2 \pi_1 (Qh_0) = (Qp_0)(Qf_0)$ and $b_2 p_2 \pi_1 (Qh_0) = (Fh_1)\beta$. Again, the latter is true because $b_2p_2\pi_1 = P_\gamma\alpha$, while, for the former, we observe that $p_2\pi_1 = P_Z\langle Qp_0, \alpha\rangle$ and that the following diagram commutes:

$$
\begin{tikzcd}
 & QX \arrow[rr, "\langle Qp_0,\alpha\rangle"] \arrow[dl, "Qp_0"'] \arrow[dd, "Qr_Z"] & & QZ \times_{FC} FA \arrow[r, equal] \arrow[dl, dashed, "q_1"] \arrow[dd, "r_{QZ}"] & \gamma^*(FA) \\
QZ \arrow[rr, equal, dashed] & & QZ & & \\
 & QP_ZX \arrow[ul] \arrow[rr, "P_Z\langle Qp_0,\alpha\rangle"'] \arrow[d, shift right, "Qs_Z"'] \arrow[d, shift left, "Qt_Z"] & & QZ \times_{FC} FP_CA \arrow[ul, dashed, "b_1"] \arrow[d, shift right, "s_{QZ}"'] \arrow[d, shift left, "t_{QZ}"] & \\
 & QX \arrow[uul, bend left, "Qp_0"] \arrow[rr, "\langle Qp_0,\alpha\rangle"'] & & QZ \times_{FC} FA \arrow[r, equal] \arrow[uul, dashed, bend right, "q_1"'] & \gamma^*(FA)
\end{tikzcd}
$$

by construction of $P_Z\langle Qp_0, \alpha\rangle$ in $\mathcal{D}(QZ)$ (see Theorem A.9 and Remark A.10). Hence we deduce that $b_1p_2\pi_1 = b_1P_Z\langle Qp_0, \alpha\rangle = (QP_ZX \to QZ) = (Qp_0)(Qs_Z)$. Therefore we conclude that $b_1\pi_2\pi_1(Qh_0) = (Qp_0)(Qs_Z)(Qh_0) = (Qp_0)(Qf_0)$. The equality ♣ holds.

Secondly, we observe that the equality:

$$t_{\gamma^*(FA)}\pi_2(Qh_0) \overset{\spadesuit}{=} Qg_0$$

holds as well, as $t_{\gamma^*(FA)}\pi_2 = Qt_Z$ (see Remark 3.16). By ♣ and ♠ we conclude that $\pi_2(Qh_0)$ is a homotopy $\Gamma_{\langle Qp_0,\alpha\rangle}\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle \simeq_{\gamma^*(FA)} Qg_0$.

With respect to the third pullback, observe that $\Gamma_{\langle Qp_0,\alpha\rangle}\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle = s_{\gamma^*(FA)}\pi_2(Qh_0)$, because $\pi_2(Qh_0)$ is a homotopy, hence the arrow:

$$QY \xrightarrow{\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, \pi_2(Qh_0)\rangle} QP_ZX$$

exists and it is equal to $Qh_0$, as we proved that $\pi_1(Qh_0) = \langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle$.

*If.* Viceversa, let us assume that there are homotopies:

$$h_1 \colon f_1 \simeq_C g_1 \text{ and } h \colon \Gamma_{\langle Qp_0,\alpha\rangle}\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle \simeq_{\gamma^*(FA)} Qg_0$$

such that the arrow:

$$QY \xrightarrow{\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle} QP_ZX$$

is the image through $Q$ of a homotopy $f_0 \simeq_Z g_0$. At first, let us observe that the arrow $\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle$ actually exists, as $h$ is required to be a homotopy. Moreover we observe that:

$$\begin{aligned}
(Qs_Z)\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle &= p_1\pi_1\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle \\
&= p_1\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle = Qf_0
\end{aligned}$$

and that:

$$\begin{aligned}
(Qt_Z)\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle &= t_{\gamma^*(FA)}\pi_2\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle \\
&= t_{\gamma^*(FA)}h = Qg_0
\end{aligned}$$

hence $\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle$ is a homotopy $Qf_0 \simeq_{QZ} Qg_0$. Finally:

$$\begin{aligned} P_\gamma\alpha(Qh_0) &= P_\gamma\alpha\langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle \\ &= b_2 p_2 \pi_1 \langle\langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle, h\rangle \\ &= b_2 p_2 \langle Qf_0, \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle\rangle \\ &= b_2 \langle (Qp_0)(Qf_0), (Fh_1)\beta\rangle = (Fh_1)\beta \end{aligned}$$

hence $(h_0, h_1)$ is an arrow $(Y, B, \beta) \to (P_Z X, P_C A, P_\gamma \alpha)$ and we are done. Q.E.D.

## 3.3 Homotopy Natural Numbers in the Generalised Gluing

As mentioned in the Introduction and in Chapter 2, this last section is about (strong) homotopy natural numbers. We prove that, if both the domain of a given exact functor between path category and the domain of a given fibred path category over its codomain have the (strong) homotopy natural numbers, then the generalised gluing has the (strong) homotopy natural numbers as well. The second result, together with its proof, is a generalisation of the corresponding one contained in [12].

**Theorem 3.19.** *Let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$ between path categories and let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. If $\mathcal{C}$ and $\mathcal{E}$ have the homotopy natural numbers object, then* $\mathrm{GL}(F, Q)$ *has the homotopy natural numbers object as well.*

*Proof.* Let $(\mathbb{N}, 0, S)$ be a homotopy natural numbers object in $\mathcal{E}$ and let $(\mathbb{N}', 0', S')$ be a homotopy natural numbers object in $\mathcal{C}$. By Proposition 3.12 it is the case that $(Q\mathbb{N}, Q0, QS)$ is a homotopy natural numbers object in $\mathcal{D}$. By Proposition 2.3, there is, unique up to homotopy, an arrow $Q\mathbb{N} \xrightarrow{f} F\mathbb{N}'$ such that the following:

$$\begin{tikzcd} 1 \arrow[r, "Q0"] \arrow[dr, "F0'"'] & Q\mathbb{N} \arrow[r, "QS"] \arrow[d, "f"] & Q\mathbb{N} \arrow[d, "f"] \\ & F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}' \end{tikzcd}$$

commutes up to homotopy. By Proposition A.5, it is the case that $f = (Q\mathbb{N} \xrightarrow{w} X \xrightarrow{\alpha} F\mathbb{N}')$ for some section $w$ of an acyclic fibration $X \xrightarrow{l} Q\mathbb{N}$ and some fibration $\alpha$ (of $\mathcal{D}$). Hence the following diagram:

$$\begin{tikzcd} & Q\mathbb{N} \arrow[r, "QS"] \arrow[d, shift left, "w"] & Q\mathbb{N} \arrow[d, shift left, "w"] \\ 1 \arrow[ur, bend left, "Q0"] \arrow[r, "w(Q0)"] \arrow[dr, bend right, "F0'"'] & X \arrow[r, "w(QS)l"] \arrow[u, shift left, "l"] \arrow[d, "\alpha"] & X \arrow[u, shift left, "l"] \arrow[d, "\alpha"] \\ & F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}' \end{tikzcd}$$

commutes up to homotopy. By Theorem A.11 and being $l$ a fibration, there are arrows

$1 \xrightarrow{x'} X$ and $X \xrightarrow{f'} X$ such that the diagram:

$$\begin{tikzcd} & Q\mathbb{N} \arrow[r, "QS"] & Q\mathbb{N} \\ 1 \arrow[ur, bend left, "Q0"] \arrow[r, "x'"] & X \arrow[u, "l"] \arrow[r, "f'"] & X \arrow[u, "l"] \end{tikzcd}$$

commutes and $x' \simeq w(Q0)$ and $f' \simeq w(QS)l$. By Remark 3.5 there is unique an arrow $l^+\mathbb{N} \xrightarrow{l^+S} l^+\mathbb{N}$ such that the diagram:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\ l^+\mathbb{N} \arrow[u, "l^+"] \arrow[r, "l^+S"] & l^+\mathbb{N} \arrow[u, "l^+"] \end{tikzcd}$$

commutes and $Q(l^+S) = f'$. Moreover, as $l^+$ is cartesian, there is unique an arrow $1 \xrightarrow{y'} l^+\mathbb{N}$ such that the following:

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\ 1 \arrow[ur, "0"] \arrow[r, "y'"] & l^+\mathbb{N} \arrow[u, "l^+"] \arrow[r, "l^+S"] & l^+\mathbb{N} \arrow[u, "l^+"] \end{tikzcd}$$

commutes and $Qy' = x'$. By Remark 3.8 and being $l$ a weak equivalence, it is the case that $l^+$ is a weak equivalence as well. Hence, by Proposition 2.4, it holds that $(l^+\mathbb{N}, y', l^+S)$ is a homotopy natural numbers object of $\mathcal{E}$ and $(X, x', f') = Q(l^+\mathbb{N}, y', l^+S)$ is a homotopy natural numbers object by Proposition 3.12. Now, as $x' \simeq w(Q0)$ and $f' \simeq w(QS)l$, it is the case that the diagram:

$$\begin{tikzcd} 1 \arrow[r, "x'"] \arrow[dr, bend right, "F0'"'] & X \arrow[r, "f'"] \arrow[d, "\alpha"] & X \arrow[d, "\alpha"] \\ & F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}' \end{tikzcd}$$

still commutes up to homotopy. Hence, again, by Theorem A.11 and being $\alpha$ a fibration, there are arrows $1 \xrightarrow{x} X$ and $X \xrightarrow{f} X$ such that:

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[dr, bend right, "F0'"'] & X \arrow[r, "f"] \arrow[d, "\alpha"] & X \arrow[d, "\alpha"] \\ & F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}' \end{tikzcd}$$

commutes and homotopies $h\colon Qy' = x' \simeq x$ and $k\colon Ql^+S = f' \simeq f$. By Corollary 3.11 there are homotopies $y' \simeq u$ and $l^+S \simeq \Sigma$ for some arrows $1 \xrightarrow{u} l^+\mathbb{N}$ and $l^+\mathbb{N} \xrightarrow{\Sigma} l^+\mathbb{N}$ such that $Qu = x$ and $Q\Sigma = f$. In particular, by Proposition 2.6, it is the case that $(\mathbb{N}, x, f)$ and $(l^+\mathbb{N}, u, \Sigma)$ are homotopy natural numbers objects and $Q(l^+\mathbb{N}, u, \Sigma) = (\mathbb{N}, x, f)$. We conclude that there is a commutative diagram:

$$\begin{tikzcd} 1 \arrow[r, "Qu"] \arrow[d, equal] & Ql^+\mathbb{N} \arrow[r, "Q\Sigma"] \arrow[d, "\alpha"] & Ql^+\mathbb{N} \arrow[d, "\alpha"] \\ 1 \arrow[r, "F0'"] & F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}' \end{tikzcd}$$

where $\alpha$ is a fibration.

We are going to prove that the object $(l^+\mathbb{N}, \mathbb{N}^+, \alpha)$ of $\mathrm{GL}(F,Q)$ together with the arrows $(u, 0')$ and $(\Sigma, S')$ is a homotopy natural numbers object of $\mathrm{GL}(F,Q)$. Let us consider a diagram:

$$\begin{tikzcd}
(1,1,1_1) \arrow[r, "{(u,0')}"] \arrow[dr, "{(y,b)}"'] & (l^+\mathbb{N}, \mathbb{N}', \alpha) \arrow[r, "{(\Sigma,S')}"] & (l^+\mathbb{N}, \mathbb{N}', \alpha) \\
& (Y,B,\beta) \arrow[r, "{(S_1,S_2)}"] & (Y,B,\beta)
\end{tikzcd}$$

in $\mathrm{GL}(F,Q)$. Being $(l^+\mathbb{N}, u, \Sigma)$ and $(\mathbb{N}', 0', S')$ there are arrows $l^+\mathbb{N} \xrightarrow{a_1} Y$ and $\mathbb{N}' \xrightarrow{a_2} B$, unique up to homotopy, such that:

$$\begin{tikzcd}
1 \arrow[r, "u"] \arrow[dr, "y"'] & l^+\mathbb{N} \arrow[r, "\Sigma"] \arrow[d, "a_1"] & l^+\mathbb{N} \arrow[d, "a_1"] \\
& Y \arrow[r, "S_1"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "0'"] \arrow[dr, "b"'] & \mathbb{N}' \arrow[r, "S'"] \arrow[d, "a_2"] & \mathbb{N}' \arrow[d, "a_2"] \\
& B \arrow[r, "S_2"] & B
\end{tikzcd}$$

commute up to homotopy. Hence the following homotopies follow:

1. $(Ql^+\mathbb{N} \xrightarrow{Q\Sigma} Ql^+\mathbb{N} \xrightarrow{\alpha} F\mathbb{N}' \xrightarrow{Fa_2} FB) \simeq (Ql^+\mathbb{N} \xrightarrow{\alpha} F\mathbb{N}' \xrightarrow{Fa_2} FB \xrightarrow{FS_2} FB)$. Indeed:
$$\begin{aligned}(Fa_2)\alpha(Q\Sigma) &= (Fa_2)(FS')\alpha \\ &\simeq (FS_2)(Fa_2)\alpha\end{aligned}$$
because $(\Sigma, S')$ is an arrow of $\mathrm{GL}(F,Q)$.

2. $(Ql^+\mathbb{N} \xrightarrow{Q\Sigma} Ql^+\mathbb{N} \xrightarrow{Qa_1} QY \xrightarrow{\beta} FB) \simeq (Ql^+\mathbb{N} \xrightarrow{Qa_1} QY \xrightarrow{\beta} FB \xrightarrow{FS_2} FB)$. Indeed:
$$\begin{aligned}\beta(Qa_1)(Q\Sigma) &\simeq \beta(QS_1)(Qa_1) \\ &= (QS_2)\beta(Qa_1)\end{aligned}$$
because $(S_1, S_2)$ is an arrow of $\mathrm{GL}(F,Q)$.

3. $(1 \xrightarrow{Qu} Ql^+\mathbb{N} \xrightarrow{Qa_1} QY \xrightarrow{\beta} FB) \simeq (1 \xrightarrow{Qy} QY \xrightarrow{\beta} FB) = (1 \xrightarrow{Fb} FB)$, because $(y,b)$ is an arrow of $\mathrm{GL}(F,Q)$.

4. $(1 \xrightarrow{Qu} Ql^+\mathbb{N} \xrightarrow{\alpha} F\mathbb{N}' \xrightarrow{Fa_2} FB) = (1 \xrightarrow{F0'} F\mathbb{N}' \xrightarrow{Fa_2} FB) \simeq (1 \xrightarrow{Fb} FB)$, because $(u, 0')$ is an arrow of $\mathrm{GL}(F,Q)$.

In other words, the following diagrams:

$$\begin{tikzcd}
1 \arrow[r, "Qu"] \arrow[dr, "Fb"'] & Ql^+\mathbb{N} \arrow[r, "Q\Sigma"] \arrow[d, "\beta(Qa_1)"] & Ql^+\mathbb{N} \arrow[d, "\beta(Qa_1)"] \\
& FB \arrow[r, "FS_2"'] & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "Qu"] \arrow[dr, "Fb"'] & Ql^+\mathbb{N} \arrow[r, "Q\Sigma"] \arrow[d, "(Fa_2)\alpha"] & Ql^+\mathbb{N} \arrow[d, "(Fa_2)\alpha"] \\
& FB \arrow[r, "FS_2"'] & FB
\end{tikzcd}$$

commute up to homotopy and, since $(Ql^+\mathbb{N}, Q\Sigma, Qu)$ is homotopy natural numbers object, we deduce that the following diagram:

$$\begin{tikzcd}
Ql^+\mathbb{N} \arrow[r, "\alpha"] \arrow[d, "Qa_1"'] & F\mathbb{N}' \arrow[d, "Fa_2"] \\
QY \arrow[r, "\beta"] & FB
\end{tikzcd}$$

needs to commute up to homotopy. Hence, being $\beta$ a fibration and by Theorem A.11, there is $Ql^+\mathbb{N} \xrightarrow{q} QY$ such that:

$$\begin{tikzcd} Ql^+\mathbb{N} \arrow[r, "\alpha"] \arrow[d, "q"'] & F\mathbb{N}' \arrow[d, "Fa_2"] \\ QY \arrow[r, "\beta"] & FB \end{tikzcd}$$

commutes and a homotopy $Qa_1 \simeq q$. Again, by the homotopy lifting property (Corollary 3.11), there is $l^+\mathbb{N} \xrightarrow{a} Y$ such that $a_1 \simeq a$ and $Qa = q$. Hence $(a, a_2)$ is an arrow $(l^+\mathbb{N}, \mathbb{N}', \alpha) \to (Y, B, \beta)$ and the diagrams:

$$\begin{tikzcd} 1 \arrow[r, "u"] \arrow[dr, "y"'] & l^+\mathbb{N} \arrow[r, "\Sigma"] \arrow[d, "a"] & l^+\mathbb{N} \arrow[d, "a"] \\ & Y \arrow[r, "S_1"] & Y \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "0'"] \arrow[dr, "b"'] & \mathbb{N}' \arrow[r, "S'"] \arrow[d, "a_2"] & \mathbb{N}' \arrow[d, "a_2"] \\ & B \arrow[r, "S_2"] & B \end{tikzcd}$$

still commute up to homotopy. Let $h_0 : au \simeq y$ and let $h_1 : a_2 0' \simeq b$. Hence it is the case that $(P\alpha)(Qh_0) \simeq (Fh_1)(1_1)\colon 1 \to FP\mathbb{N}'$. Therefore, being $P\alpha$ a fibration and by Theorem A.11 and Corollary 3.11, there is an arrow $h_0'$ such that $(P\alpha)(Qh_0') = (Fh_1)(1_1)$ together with an arrow $h_0''\colon 1 \to P(Pl^+\mathbb{N})$ such that $h_0''\colon h_0 \simeq h_0'$. Now, we observe that $P(Pl^+\mathbb{N})$ is still a path object over $l^+\mathbb{N}$ together with the arrows $l^+\mathbb{N} \xrightarrow{r} Pl^+\mathbb{N} \xrightarrow{r'} PPl^+\mathbb{N}$ and $PPl^+\mathbb{N} \xrightarrow{\langle ss', ts'\rangle} l^+\mathbb{N} \times l^+\mathbb{N}$, where $(PPl^+\mathbb{N}, r', \langle s', t'\rangle)$ a path object over $Pl^+\mathbb{N}$. Indeed, $\langle ss', ts'\rangle r'r = \langle ss'r'r, ts'r'r\rangle = \langle sr, tr\rangle = \langle 1_{l+\mathbb{N}}, 1_{l+\mathbb{N}}\rangle$. Moreover, as $\langle Qs', Qt'\rangle$ is a fibration, it is the case that $QPPl^+\mathbb{N} \xrightarrow{(P\alpha') := (P\alpha)(Qt')} FP\mathbb{N}'$ is still a path object over over $(l^+\mathbb{N}, \mathbb{N}', \alpha)$ in $\mathrm{GL}(F, Q)$. Finally, we observe that $\langle ss', ts'\rangle h_0'' = \langle s, t\rangle h_0 = \langle au, y\rangle$ and moreover $(P\alpha')(Qh_0'') = (P\alpha)(Qt')(Qh_0'') = (P\alpha)(Qh_0') = (Fh_1)(1_1)$. This concludes that $(h_0'', h_1)$ is an arrow of $\mathrm{GL}(F, Q)$ and a homotopy $(au, a_2 0') \simeq (y, b)$. The exact same argument allows us to conclude that $(a\Sigma, a_2 S') \simeq (S_1 a, S_2 a_2)$ in $\mathrm{GL}(F, Q)$. Hence $(a, a_2)$ is an arrow $(l^+\mathbb{N}, \mathbb{N}', \alpha)$ such that:

$$\begin{tikzcd} (1,1,1_1) \arrow[r, "{(u,0')}"] \arrow[dr, "{(y,b)}"'] & (l^+\mathbb{N}, \mathbb{N}', \alpha) \arrow[r, "{(\Sigma,S')}"] \arrow[d, "{(a,a_2)}"] & (l^+\mathbb{N}, \mathbb{N}', \alpha) \arrow[d, "{(a,a_2)}"] \\ & (Y, B, \beta) \arrow[r, "{(S_1,S_2)}"] & (Y, B, \beta) \end{tikzcd}$$

commutes up to homotopy. Q.E.D.

**Conjecture 3.20.** *Let $F$ be an exact functor $\mathcal{C} \to \mathcal{D}$ between path categories and let $Q\colon \mathcal{E} \to \mathcal{D}$ be a fibred path category over $\mathcal{D}$. If $\mathcal{C}$ and $\mathcal{E}$ have the strong homotopy natural numbers object, then* $\mathrm{GL}(F, Q)$ *has the strong homotopy natural numbers object as well.*

*Proof - Work in progress.* Let $(\mathbb{N}, 0, S)$ be a strong homotopy natural numbers object in $\mathcal{E}$ and let $(\mathbb{N}', 0', S')$ be a strong homotopy natural numbers object in $\mathcal{C}$. By Proposition 3.12 it is the case that $(Q\mathbb{N}, Q0, QS)$ is a strong homotopy natural numbers object in $\mathcal{D}$. By Proposition 2.3, there is, unique up to homotopy, an arrow $Q\mathbb{N} \xrightarrow{f} F\mathbb{N}'$ such that the following:

$$\begin{tikzcd}
1 \arrow[r, "Q0"] \arrow[dr, "F0'"'] & Q\mathbb{N} \arrow[r, "QS"] \arrow[d, "f"] & Q\mathbb{N} \arrow[d, "f"] \\
& F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}'
\end{tikzcd}$$

commutes up to homotopy. By Proposition A.5, it is the case that $f = (Q\mathbb{N} \xrightarrow{w} X \xrightarrow{\gamma} F\mathbb{N}')$ for some section $w$ of an acyclic fibration $X \xrightarrow{l} Q\mathbb{N}$ and some fibration $\gamma$ (of $\mathcal{D}$). Hence the following diagram:

$$\begin{tikzcd}
& Q\mathbb{N} \arrow[r, "QS"] \arrow[d, shift left, "w"] & Q\mathbb{N} \arrow[d, shift left, "w"] \\
1 \arrow[ur, bend left, "Q0"] \arrow[r, "w(Q0)"] \arrow[dr, bend right, "F0'"'] & X \arrow[u, shift left, "l"] \arrow[r, "w(QS)l"] \arrow[d, "\gamma"] & X \arrow[u, shift left, "l"] \arrow[d, "\gamma"] \\
& F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}'
\end{tikzcd}$$

commutes up to homotopy. By Theorem A.11 and being $l$ a fibration, there are arrows $1 \xrightarrow{x'} X$ and $X \xrightarrow{f'} X$ such that the diagram:

$$\begin{tikzcd}
& Q\mathbb{N} \arrow[r, "QS"] & Q\mathbb{N} \\
1 \arrow[ur, bend left, "Q0"] \arrow[r, "x'"] & X \arrow[u, "l"] \arrow[r, "f'"] & X \arrow[u, "l"]
\end{tikzcd}$$

commutes and $x' \simeq w(Q0)$ and $f' \simeq w(QS)l$. By Remark 3.5 there is unique an arrow $l^+\mathbb{N} \xrightarrow{l^+S} l^+\mathbb{N}$ such that the diagram:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "S"] & \mathbb{N} \\
l^+\mathbb{N} \arrow[u, "l^+"] \arrow[r, "l^+S"] & l^+\mathbb{N} \arrow[u, "l^+"]
\end{tikzcd}$$

commutes and $Q(l^+S) = f'$. Moreover, as $l^+$ is cartesian, there is unique an arrow $1 \xrightarrow{y'} l^+\mathbb{N}$ such that the following:

$$\begin{tikzcd}
& \mathbb{N} \arrow[r, "S"] & \mathbb{N} \\
1 \arrow[ur, "0"] \arrow[r, "y'"] & l^+\mathbb{N} \arrow[u, "l^+"] \arrow[r, "l^+S"] & l^+\mathbb{N} \arrow[u, "l^+"]
\end{tikzcd}$$

commutes and $Qy' = x'$. By Remark 3.8 and being $l$ an acyclic fibration, it is the case that $l^+$ is an acyclic fibration as well. Hence, being $(\mathbb{N}, 0, S)$ a strong homotopy natural numbers object, there is a section $\mathbb{N} \xrightarrow{u} l^*\mathbb{N}$ of $l^+$ such that $u0 \simeq_{\mathbb{N}} y'$ and $uS \simeq_{\mathbb{N}} (l^+S)u$. In particular, it is the case that $y' \simeq u0$ and $l^+S \simeq (l^+S)ul^+ \simeq uSl^+$. Therefore $(l^+\mathbb{N}, y, l^+S)$ is a strong natural numbers object, by Proposition 2.6 and being $(l^+\mathbb{N}, u0, uSl^+)$ a strong homotopy natural numbers object by Proposition 2.7. Moreover $(X, x', f') = Q(l^+\mathbb{N}, y', l^+S)$ is a homotopy natural numbers object by Proposition 3.12. Now, as $x' \simeq w(Q0)$ and $f' \simeq w(QS)l$, it is the case that the diagram:

$$\begin{tikzcd}
1 \arrow[r, "x'"] \arrow[dr, bend right, "F0'"'] & X \arrow[r, "f'"] \arrow[d, "\gamma"] & X \arrow[d, "\gamma"] \\
& F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}'
\end{tikzcd}$$

still commutes up to homotopy. Hence, again, by Theorem A.11 and being $\gamma$ a fibration, there are arrows $1 \xrightarrow{x} X$ and $X \xrightarrow{f} X$ such that:

$$\begin{tikzcd}
1 \arrow[r, "x"] \arrow[dr, bend right, "F0'"'] & X \arrow[r, "f"] \arrow[d, "\gamma"] & X \arrow[d, "\gamma"] \\
& F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}'
\end{tikzcd}$$

commutes and homotopies $h\colon Qy' = x' \simeq x$ and $k\colon Ql^+S = f' \simeq f$. By Corollary 3.11 there are homotopies $y' \simeq u$ and $l^+S \simeq \Sigma$ for some arrows $1 \xrightarrow{u} l^+\mathbb{N}$ and $l^+\mathbb{N} \xrightarrow{\Sigma} l^+\mathbb{N}$ such that $Qu = x$ and $Q\Sigma = f$. In particular, by Proposition 2.6, it is the case that $(\mathbb{N}, x, f)$ and $(l^+\mathbb{N}, u, \Sigma)$ are strong homotopy natural numbers objects and $Q(l^+\mathbb{N}, u, \Sigma) = (\mathbb{N}, x, f)$. We conclude that there is a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r, "Qu"] \arrow[d, equal] & QZ \arrow[r, "Q\Sigma"] \arrow[d, "\gamma"] & QZ \arrow[d, "\gamma"] \\
1 \arrow[r, "F0'"] & F\mathbb{N}' \arrow[r, "FS'"] & F\mathbb{N}'
\end{tikzcd}$$

where $\gamma$ is a fibration and $(Z := l^+\mathbb{N}, u, \Sigma)$ and $(\mathbb{N}', S', 0')$ are strong homotopy natural numbers objects.

We are going to prove that the object $(Z, \mathbb{N}', \gamma)$ of $\mathrm{GL}(F, Q)$ together with the arrows $(u, 0')$ and $(\Sigma, S')$ is a homotopy natural numbers object of $\mathrm{GL}(F, Q)$. Let us consider a commutative diagram:

$$\begin{tikzcd}
(1,1,1_1) \arrow[r, "{(u,0')}"] \arrow[dr, "{(y,b)}"'] & (Z,\mathbb{N}',\gamma) \arrow[r, "{(\Sigma,S')}"] & (Z,\mathbb{N}',\gamma) \\
& (Y,B,\beta) \arrow[r, "{(S_1,S_2)}"] \arrow[u, "{(p_0,p_1)}"'] & (Y,B,\beta) \arrow[u, "{(p_0,p_1)}"']
\end{tikzcd}$$

in $\mathrm{GL}(F, Q)$, where $(p_0, p_1)$ is a fibration. Being $(\mathbb{N}', 0', S')$ a strong homotopy natural numbers object there is unique up to homotopy a section $\mathbb{N}' \xrightarrow{a_1} B$ of $p_1$ such that there are homotopies $h_1\colon b \simeq_{\mathbb{N}'} a_1 0'$ and $h_1'\colon S_2 a_1 \simeq_{\mathbb{N}'} a_1 S'$.

Let us consider the following diagram:

$$\begin{tikzcd}
\Phi \arrow[r, "\pi_Y"] \arrow[d, "\pi_Z"'] & QY \arrow[d, "{\langle Qp_0, \beta\rangle}"] & \\
QZ \arrow[r, "{\langle 1_{QX}, (Fa_1)\gamma\rangle}"] & \gamma^*(FB) \arrow[r, "q_2"] \arrow[d, "q_1"'] & FB \arrow[d, "Fp_1"] \\
& QZ \arrow[r, "\gamma"] & F\mathbb{N}'
\end{tikzcd}$$

whose two squares are pullbacks and observe that $\pi_Z = 1_{QX}\pi_Z = (Qp_0)\pi_Y$. Let $(P_{\mathbb{N}'}B \to \mathbb{N}', r_{\mathbb{N}'}, \langle s_{\mathbb{N}'}, t_{\mathbb{N}'}\rangle)$ be a fibred path object of $B$ over $\mathbb{N}'$. Hence, as usual, we get a fibred path object of $\gamma^*(FB)$ over $QZ$, as in the following commutative diagram:

$$\begin{tikzcd}
 & \gamma^*(FB) \arrow[rr, "q_2"] \arrow[dl, "q_1"'] \arrow[dd, dashed, "r_{QZ}"] & & FB \arrow[dl, dashed, "Fp_1"'] \arrow[dd, "Fr_{\mathbb{N}'}"] \\
QZ \arrow[rr, dashed, "\gamma"] & & F\mathbb{N}' & \\
 & QZ \times_{F\mathbb{N}'} FP_{\mathbb{N}'}B \arrow[ul, "b_1"] \arrow[rr, "b_2"'] \arrow[d, shift right, "s_{QZ}"'] \arrow[d, shift left, "t_{QZ}"] & & FP_{\mathbb{N}'}B \arrow[ul, dashed, "\bullet"'] \arrow[d, shift right, "Fs_{\mathbb{N}'}"'] \arrow[d, shift left, "Ft_{\mathbb{N}'}"] \\
 & \gamma^*(FB) \arrow[rr, "q_2"'] \arrow[uul, bend left, "q_1"] & & FB \arrow[uul, dashed, bend left, "Fp_1"]
\end{tikzcd}$$

(see Remark 3.16).

Observe that the arrow $QY \xrightarrow{\langle Qp_0,\beta\rangle} \gamma^*(FB)$ is fibred over over $QZ$, being $\langle Qp_0, \beta\rangle$ and $q_1$ fibrations of $\mathcal{D}$. Hence we can consider a transport structure $P_{\langle Qp_0,\beta\rangle} \xrightarrow{\Gamma_{\langle Qp_0,\beta\rangle}} QY$ in $\mathcal{D}(QZ)$ for the arrow $\langle Qp_0, \beta\rangle$ of $\mathcal{D}(QZ)$.

We observe that the following square:

$$\begin{tikzcd}
1 \arrow[r, "Qy"] \arrow[d, "Qu"'] & QY \arrow[d, "\beta"] \\
QZ \arrow[r, "(Fa_1)\gamma"] & FB
\end{tikzcd}$$

commutes up to the homotopy $Fh_1\colon \beta(Qy) = Fb \simeq_{\mathbb{FN}'} (Fa_1)(F0') = (Fa_1)\gamma(Qu)$. Let us observe that, with respect to the pullback:

$$\begin{tikzcd}
QZ \times_{F\mathbb{N}'} FP_{\mathbb{N}'}B \arrow[r, "b_2"] \arrow[d, "b_1"'] & FP_{\mathbb{N}'}B \arrow[d, "\bullet"] \\
QZ \arrow[r, "\gamma"] & F\mathbb{N}'
\end{tikzcd}$$

it is the case that: $\bullet(Fh_1) = (Fp_1)(Ft_{\mathbb{N}'})(Fh_1) = (Fp_1)(Fa_1)(F0') = F0' = \gamma(Qu)$. Hence the arrow $1 \xrightarrow{\langle Qu,Fh_1\rangle} QZ \times_{F\mathbb{N}'} FP_{\mathbb{N}'}B$ exists. We claim that it is a homotopy:

$$\langle Qp_0, \beta\rangle(Qy) \simeq_{QZ} \langle 1_{QX}, (Fa_1)\gamma\rangle(Qu).$$

At first we verify that:

$$s_{QZ}\langle Qu, Fh_1\rangle = \langle Qp_0, \beta\rangle(Qy),$$

that is, $q_1 s_{QZ}\langle Qu, Fh_1\rangle = (Qp_0)(Qy)$ and $q_2 s_{QZ}\langle Qu, Fh_1\rangle = \beta(Qy)$. The former is true because $q_1 s_{QZ} = b_1$ and $Qu = (Qp_0)(Qy)$, while the latter holds because $q_2 s_{QZ} = (Fs_{\mathbb{N}'})b_2$ and $(Fs_{\mathbb{N}'})(Fh_1) = \beta(Qy)$.

Secondly, we need to verify that:

$$t_{QZ}\langle Qu, Fh_1\rangle = \langle 1_{QX}, (Fa_1)\gamma\rangle(Qu),$$

that is, $q_1 t_{QZ}\langle Qu, Fh_1\rangle = Qu$ and $q_2 t_{QZ}\langle Qu, Fh_1\rangle = (Fa_1)\gamma(Qu)$. The former is true because $q_1 t_{QZ} = b_1$. For the latter, observe that $q_2 t_{QZ} = (Ft_{\mathbb{N}'})b_2$, hence $q_2 t_{QZ}\langle Qu, Fh_1\rangle = (Ft_{\mathbb{N}'})(Fh_1) = (Fa_1)\gamma(Qu)$.

We proved that the following square:

$$\begin{tikzcd} 1 \arrow[r, "Qy"] \arrow[d, "Qu"'] & QY \arrow[d, "\langle Qp_0, \beta\rangle"] \\ QZ \arrow[r, "\langle 1_{QZ}, (Fa_1)\gamma\rangle"] & \gamma^*(FB) \end{tikzcd}$$

commutes up to fibrewise homotopy $\langle Qu, Fh_1\rangle$ over $QZ$. Therefore, by Theorem A.11 applied to $\mathcal{D}(QZ)$ and being $\langle Qp_0, \beta\rangle$ a fibration of $\mathcal{D}(QZ)$, the following diagram:

$$\begin{tikzcd} 1 \arrow[r, "Qy'"] \arrow[d, "Qu"'] & QY \arrow[d, "\langle Qp_0, \beta\rangle"] \\ QZ \arrow[r, "\langle 1_{QZ}, (Fa_1)\gamma\rangle"] & \gamma^*(FB) \end{tikzcd}$$

strictly commutes, being $Qy' = (1 \xrightarrow{\Gamma_{\langle Qp_0,\beta\rangle}\langle Qy, \langle Qu, Fh_1\rangle\rangle} QY)$ and being $1 \xrightarrow{y'} Y$ such that $y' \simeq_Z y$ (we used Corollary 3.11). Therefore, there is an arrow (♠):

$$1 \xrightarrow{\langle Qu, Qy'\rangle} \Phi$$

such that $\pi_Z\langle Qu, Qy'\rangle = Qu$ and $\pi_Y\langle Qu, Qy'\rangle = Qy' = \Gamma_{\langle Qp_0,\beta\rangle}\langle Qy, \langle Qu, Fh_1\rangle\rangle$.

We observe that the following square:

$$\begin{tikzcd} \Phi \arrow[r, "(QS_1)\pi_Y"] \arrow[d, "(Q\Sigma)\pi_Z"'] & QY \arrow[d, "\beta"] \\ QZ \arrow[r, "(Fa_1)\gamma"] & FB \end{tikzcd}$$

commutes up to fibred homotopy:

$$\begin{aligned} (Fh_1')\gamma\pi_Z \colon \beta(QS_1)\pi_Y &= (FS_2)\beta\pi_Y \\ &= (FS_2)(Fa_1)\gamma\pi_Z \\ &\simeq_{F\mathbb{N}'} (Fa_1)(FS')\gamma\pi_Z \\ &= (Fa_1)\gamma(Q\Sigma)\pi_Z. \end{aligned}$$

Moreover, with respect to the pullback:

$$\begin{tikzcd} QZ \times_{F\mathbb{N}'} FP_{\mathbb{N}'}B \arrow[r, "b_2"] \arrow[d, "b_1"'] & FP_{\mathbb{N}'}B \arrow[d, "\bullet"] \\ QZ \arrow[r, "\gamma"] & F\mathbb{N}' \end{tikzcd}$$

it is the case that $\bullet(Fh_1')\gamma\pi_Z = (Fp_1)(Ft_{\mathbb{N}'})(Fh_1')\gamma\pi_Z = (Fp_1)(Fa_1)\gamma(Q\Sigma)\pi_Z = \gamma(Q\Sigma)\pi_Z$. Hence the arrow:

$$\Phi \xrightarrow{\langle Q\Sigma, (Fh_1')\gamma\rangle\pi_Z} QZ \times_{F\mathbb{N}'} FP_{\mathbb{N}'}B$$

exists and we are going to prove that it is a homotopy:

$$\langle Qp_0, \beta\rangle(QS_1)\pi_Y \simeq_{F\mathbb{N}'} \langle 1_{QZ}, (Fa_1)\gamma\rangle(Q\Sigma)\pi_Z.$$

At first we need to verify that it is the case that $q_1 s_{QZ}\langle Q\Sigma, (Fh'_1)\gamma\rangle\pi_Z = (Qp_0)(QS_1)\pi_Y$ and that $q_2 s_{QZ}\langle Q\Sigma, (Fh'_1)\gamma\rangle\pi_Z = \beta(QS_1)\pi_Y$. The former is true, since $q_1 s_{QZ} = b_1$ and $(Q\Sigma)\pi_Z = (Q\Sigma)(Qp_0)\pi_Y = (Qp_0)(QS_1)\pi_Y$. The latter is true as well, as $q_2 s_{QZ} = Fs_{\mathbb{N}'}b_2$ and $(Fs_{\mathbb{N}'})(Fh'_1)\gamma\pi_Z = \beta(QS_1)\pi_Y$. Secondly, we verify that $q_1 t_{QZ}\langle Q\Sigma, (Fh'_1)\gamma\rangle\pi_Z = (Q\Sigma)\pi_Z$ and $q_2 t_{QZ}\langle Q\Sigma, (Fh'_1)\gamma\rangle\pi_Z = (Fa_1)\gamma(Q\Sigma)\pi_Z$. The first is true because $q_1 s_{QZ} = b_1$ and the second is true because $q_2 t_{QZ} = Ft_{\mathbb{N}'}b_2$ and $(Ft_{\mathbb{N}'})(Fh'_1)\gamma\pi_Z = (Fa_1)\gamma(Q\Sigma)\pi_Z$. We proved that the diagram:

$$\begin{tikzcd}
\Phi \arrow[r, "(QS_1)\pi_Y"] \arrow[d, "(Q\Sigma)\pi_Z"'] & QY \arrow[d, "\langle Qp_0,\beta\rangle"] \\
QZ \arrow[r, "\langle 1_{QZ},(Fa_1)\gamma\rangle"] & \gamma^*(FB)
\end{tikzcd}$$

commutes up to fibrewise homotopy $\langle Q\Sigma, (Fh'_1)\gamma\rangle\pi_Z$ over $QZ$. Therefore, by Theorem A.11 applied to $\mathcal{D}(QZ)$ and being $\langle Qp_0, \beta\rangle$ a fibration of $\mathcal{D}(QZ)$, the following diagram:

$$\begin{tikzcd}
\Phi \arrow[r, "k"] \arrow[d, "(Q\Sigma)\pi_Z"'] & QY \arrow[d, "\langle Qp_0,\beta\rangle"] \\
QZ \arrow[r, "\langle 1_{QZ},(Fa_1)\gamma\rangle"] & \gamma^*(FB)
\end{tikzcd}$$

commutes, being $k := \Gamma_{\langle Qp_0,\beta\rangle}\langle (QS_1)\pi_Y, \langle Q\Sigma, (Fh'_1)\gamma\rangle\pi_Z\rangle$. Then there is an arrow (♣):

$$\Phi \xrightarrow{\langle (Q\Sigma)\pi_Z, k\rangle} \Phi$$

such that $\pi_Z\langle (Q\Sigma)\pi_Z, k\rangle = (Q\Sigma)\pi_Z$ and $\pi_Y\langle (Q\Sigma)\pi_Z, k\rangle = k$.

Now, let us consider the commutative triangle:

$$\begin{tikzcd}
QY \arrow[rr, "\langle Qp_0,\beta\rangle"] \arrow[dr, "Qp_0"'] & & \gamma^*(FB) \arrow[dl, "q_1"] \\
& QZ &
\end{tikzcd}$$

whose arrows are fibrations. Hence, as in Remark 3.16, a path object:

$$\Big(P_{\gamma^*(FB)}QY \to \gamma^*(FB),\, r_{\gamma^*(FB)},\, \langle s_{\gamma^*(FB)}, t_{\gamma^*(FB)}\rangle\Big)$$

of $\langle Qp_0, \beta\rangle$ in $\mathcal{D}(\gamma^*(FB))$ provides a fibred path object of the object $\langle Qp_0, \beta\rangle$ of the path category $(\mathcal{D}(QZ))(q_1)$. As usual, we can apply Theorem A.9 and Remark A.10 in order to get a fibred path object of $QY$ over $QZ$ (w.r.t. $Qp_0$), that is, a path object of $QY \xrightarrow{Qp_0} QZ$ in $\mathcal{D}(QZ)$. Indeed, if the following square (#):

$$\begin{tikzcd}
P_{\langle Qp_0,\beta\rangle} \times_{QY} P_{\gamma^*(FB)}QY \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_{\gamma^*(FB)}QY \arrow[d, "s_{\gamma^*(FB)}"] \\
P_{\langle Qp_0,\beta\rangle} \arrow[r, "\Gamma_{\langle Qp_0,\beta\rangle}"] & QY
\end{tikzcd}$$

is a pullback in $\mathcal{D}(QZ)$, then:

$$QP_ZY = P_{QZ}QY = P_{\langle Qp_0,\beta\rangle} \times_{QY} P_{\gamma^*(FB)}QY$$
$$(QY \xrightarrow{Qr_Z} P_{QZ}QY) = \langle w_{\langle Qp_0,\beta\rangle}, r_{\gamma^*(FB)}\rangle \text{ [as } \Gamma_{\langle Qp_0,\beta\rangle} w_{\langle Qp_0,\beta\rangle} = 1_{QY} = s_{\gamma^*(FB)} r_{\gamma^*(FB)}]$$
$$(P_{QZ}QY \xrightarrow{Qs_Z} QY) = (P_{QZ}QY \xrightarrow{\pi_1} P_{\langle Qp_0,\beta\rangle} \xrightarrow{p_1} QY)$$
$$(P_{QZ}QY \xrightarrow{Qt_Z} QY) = (P_{QZ}QY \xrightarrow{\pi_2} P_{\gamma^*(FB)}QY \xrightarrow{t_{\gamma^*(FB)}} QY)$$

being the following diagram:

$$\begin{tikzcd}
P_{\langle Qp_0,\beta\rangle} \arrow[r, "p_2"] \arrow[d, "p_1"'] & QZ \times_{F\mathbb{N}'} FP_{\mathbb{N}'}B \arrow[d, "s_{QZ}"] \\
QY \arrow[r, "\langle Qp_0,\beta\rangle"] & \gamma^*(FB)
\end{tikzcd}$$

a pullback in $\mathcal{D}(QZ)$, and for some $(P_ZY \to Z, r_Z, \langle s_Z, t_Z\rangle)$ path object of $Y \xrightarrow{p_0} Z$ in $\mathcal{E}(Z)$, being $Q$ a fibred path category.

Moreover, let us consider the following diagram:

$$\begin{tikzcd}
 & \Phi \arrow[rr, "\pi_Y"] \arrow[dl, "\pi_Z"'] \arrow[dd, "r_{QZ}"] & & QY \arrow[dl, dashed, "\langle Qp_0,\beta\rangle"'] \arrow[dd, "r_{\gamma^*(FB)}"] \\
QZ \arrow[rr, dashed] & & \gamma^*(FB) & \\
 & P_{QZ}\Phi \arrow[ul, "\rho_1"] \arrow[rr, "\rho_2"'] \arrow[dd, shift right, "s_{QZ}"'] \arrow[dd, shift left, "t_{QZ}"] & & P_{\gamma^*(FB)}QY \arrow[ul, dashed] \arrow[dd, shift right, "s_{\gamma^*(FB)}"'] \arrow[dd, shift left, "t_{\gamma^*(FB)}"] \\
 & & & \\
 & \Phi \arrow[rr, "\pi_Y"'] \arrow[uuul, bend left, "\pi_Z"] & & QY \arrow[uuul, dashed, bend left, "\langle Qp_0,\beta\rangle"]
\end{tikzcd}$$

where the left-hand side is the pullback of right-hand side w.r.t. the arrow $QZ \xrightarrow{\langle 1_{QZ},(Fa_1)\gamma\rangle} \gamma^*(FB)$ (the dotted one in the diagram). We get a path object $P_{QZ}\Phi \xrightarrow{\rho_1} QZ$ of the arrow $\Phi \xrightarrow{\pi_Z} QZ$ in $\mathcal{D}(QZ)$.

By ♠ and ♣ the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "Qu"] \arrow[dr, "\langle Qu,Qy'\rangle"'] & QZ \arrow[r, "Q\Sigma"] & QZ \\
 & \Phi \arrow[u, "\pi_Z"'] \arrow[r, "\langle (Q\Sigma)\pi_Z,k\rangle"'] & \Phi \arrow[u, "\pi_Z"']
\end{tikzcd}$$

commutes and, being $\pi_Z$ a fibration and $(QZ, Qu, Q\Sigma)$ a strong homotopy natural numbers object, there is a section $QZ \xrightarrow{a} \Phi$ of $\pi_Z$ such that there are homotopies:

$$(1 \xrightarrow{h_0} P_{QZ}\Phi)\colon \langle Qu, Qy'\rangle \simeq_{QZ} a(Qu) \text{ and } (QZ \xrightarrow{h_0'} P_{QZ}\Phi)\colon \langle (Q\Sigma)\pi_Z, k\rangle a \simeq_{QZ} a(Q\Sigma).$$

Let $(QZ \xrightarrow{b_0} QY) := \pi_Y a$ and let us observe that $\beta b_0 = \beta\pi_Y a = (Fa_1)\gamma\pi_Z a = (Fa_1)\gamma$ and that $b_0$ is a section of $Qp_0$, as $(Qp_0)\pi_Y = \pi_Z$. Moreover, let us observe that:

1. $t_{\gamma^*(FB)}\rho_2 h_0 = \pi_Y t_{QZ} h_0 = \pi_Y a(Qu) = b_0(Qu)$
2. $s_{\gamma^*(FB)}\rho_2 h_0 = \pi_Y s_{QZ} h_0 = \pi_Y \langle Qu, Qy' \rangle = Qy' = \Gamma_{\langle Qp_0, \beta \rangle} \langle Qy, \langle Qu, Fh_1 \rangle \rangle$
3. $t_{\gamma^*(FB)}\rho_2 h_0' = \pi_Y t_{QZ} h_0' = \pi_Y a(Q\Sigma) = b_0(Q\Sigma)$
4. $s_{\gamma^*(FB)}\rho_2 h_0' = \pi_Y s_{QZ} h_0' = \pi_Y \langle (Q\Sigma)\pi_Z, k \rangle a = ka = \Gamma_{\langle Qp_0, \beta \rangle} \langle (QS_1)b_0, \langle Q\Sigma, (Fh_1')\gamma \rangle \rangle$

that is, the arrows:

$$1 \xrightarrow{\rho_2 h_0} P_{\gamma^*(FB)} QY \text{ and } QZ \xrightarrow{\rho_2 h_0'} P_{\gamma^*(FB)} QY$$

are -fibred over $\gamma^*(FB)$- homotopies $\Gamma_{\langle Qp_0,\beta\rangle} \langle Qy, \langle (Qp_0)(Qy), Fh_1 \rangle \rangle \simeq_{\gamma^*(FB)} b_0(Qu)$ and $\Gamma_{\langle Qp_0,\beta\rangle} \langle (QS_1)b_0, \langle (Qp_0)(QS_1)b_0, (Fh_1')\gamma \rangle \rangle \simeq_{\gamma^*(FB)} b_0(Q\Sigma)$ respectively (observe that $Qu = (Qp_0)(Qy)$ and $Q\Sigma = (Qp_0)(QS_1)b_0$, as it is the case that $(Q\Sigma)(Qp_0) = (Qp_0)(QS_1)$).

*This last argument has to be fixed.* By Lemma 3.17 the arrow $1 \xrightarrow{\langle \langle Qy, \langle (Qp_0)(Qy), Fh_1 \rangle \rangle, \rho_2 h_0 \rangle} QP_Z Y$ exists with respect to the pullback # and is a homotopy $Qy \simeq_{QZ} b_0(Qu)$ which agrees with $h_1$. Hence by Corollary 3.11 there is a homotopy $y \simeq_Z y'$ such that $Qy' = b_0(Qu)$ and whose image is $\langle \langle Qy, \langle (Qp_0)(Qy), Fh_1 \rangle \rangle, \rho_2 h_0 \rangle$. In particular $p_0 y' = p_0 y = u$. Analogously, there is a homotopy $(QS_1)b_0 \simeq_{QZ} b_0(Q\Sigma)$ which agrees with $h_1'$ and we can apply the same argument in order to get an arrow $(Y \xrightarrow{S_1'} Y) \simeq_Z S_1$ such that the diagram:

$$\begin{tikzcd} 1 \arrow[r, "u"] \arrow[dr, "y'"'] & Z \arrow[r, "\Sigma"] & Z \\ & Y \arrow[u, "p_0"'] \arrow[r, "S_1'"] & Y \arrow[u, "p_0"'] \end{tikzcd}$$

commutes. Being $(Z, u, \Sigma)$ a strong homotopy natural numbers object, there is a section $Z \xrightarrow{a_0} Y$ of $p_0$ such that $y' \simeq_Z a_0 u$ and $S_1' a_0 \simeq_Z a_0 \Sigma$. In particular there are homotopies $k_0 \colon y \simeq_Z a_0 u$ and $k_1 \colon a_0 \Sigma \simeq_Z S_1 a_0$ whose images are the homotopies $Qy \simeq_Q Z b_=(Qu)$ and $(QS_1)b_0 \simeq_{QZ} b_0(Q\Sigma)$ which agree with $h_1$ and $h_1'$ respectively and $Qa_0 = b_0$. By Theorem 3.18 we conclude that $(a_0, a_1)$ is an arrow $(Z, \mathbb{N}', \gamma) \to (Y, B, \beta)$ such that $(a_0, a_1)(u, 0') \simeq_{(Z,\mathbb{N}',\gamma)} (y, b)$ and $(a_0, a_1)(\Sigma, S') \simeq_{(Z,\mathbb{N}',\gamma)} (S_1, S_2)(a_0, a_1)$ and $(a_0, a_1)$ is a section of $(p_0, p_1)$. Q.E.D.

# A Appendix

## A.1 Basic Properties of Path Categories

This appendix contains every basic notion and property (without proof - they can be found in [9]) about path categories that we use in Chapter 1. Let us start with the fundamental:

**Definition A.1.** Let $\mathcal{C}$ be a path category and let $f, g$ be arrows $Y \to X$. A *homotopy from $f$ to $g$ (relative to a given path object $PX$)* is an arrow $Y \xrightarrow{h} PX$ such that $(Y \xrightarrow{h} PX \xrightarrow{\langle s,t\rangle} X \times X) = (Y \xrightarrow{\langle f,g\rangle} X \times X)$ (see 4. of Definition 1.3).

Definition A.1 does not depend on the choice of the path object $PX$, because of the following:

**Proposition A.2.** *Let $\mathcal{C}$ be a path category and let $(PX, r, \langle s,t\rangle)$ and $(P'X, r', \langle s', t'\rangle)$ be triples verifing 6. of Definition 1.3. Then there is an arrow $PX \to P'X$ commuting with $r$ and $r'$ and with $\langle s,t\rangle$ and $\langle s',t'\rangle$.*

Hence, if $f$ and $g$ are parallel arrows of a path category $\mathcal{C}$ and there is a homotopy from $f$ to $g$ (relative to some path object), then we say that *$f$ and $g$ are homotopic* and we write $f \simeq g$. Moreover the following holds:

**Proposition A.3.** *Let $\mathcal{C}$ be a path category. Then the relation $\simeq$ between parallel arrows of $\mathcal{C}$ defines a congruence on $\mathcal{C}$, that is: whenever $Y$ and $X$ are objects of $\mathcal{C}$, the relation $\simeq \subseteq \mathcal{C}(Y,X) \times \mathcal{C}(Y,X)$ is an equivalence relation on $\mathcal{C}(Y,X)$ and, whenever $f$ and $g$ are parallel arrows $Y \to X$ such that $f \simeq g$ and $k$ and $l$ are parallel arrows $Z \to Y$ such that $k \simeq l$, then it is the case that $fk \simeq gl$.*

By Proposition A.3, whenever $\mathcal{C}$ is a path category, there is a category $\mathrm{Ho}(\mathcal{C})$, called *homotopy category of* $\mathcal{C}$, whose objects are the ones of $\mathcal{C}$ and whose arrows are the equivalence classes of parallel arrows of $\mathcal{C}$ modulo the relation $\simeq$. In particular, one can prove that:

**Theorem A.4.** *If $\mathcal{C}$ is a path category, then an arrow of $\mathcal{C}$ represents an isomorphism of* $\mathrm{Ho}(\mathcal{C})$ *(an arrow of $\mathcal{C}$ with this property will be called* homotopy equivalence*) if and only if it is a weak equivalence.*

Now, let us state the following two basic facts:

**Proposition A.5.** *Let $\mathcal{C}$ be a path category. Then for every arrow $Y \xrightarrow{f} X$ of $\mathcal{C}$ there are a fibration $P_f \xrightarrow{p_f} X$ and a section $Y \xrightarrow{w_f} P_f$ of an acyclic fibration such that $f = p_f w_f$. This factorization is obtained as follows. Let us consider the pullback:*

$$\begin{tikzcd}
P_f \arrow[r, "p_2"] \arrow[d, "p_1"'] & PX \arrow[d, "s"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

*of $s$ along $f$, being $(PX, r, \langle s,t\rangle)$ a triple satisfying 6. of Definition 1.3 (it exists because of 2. of Definition 1.3). Then one defines $(P_f \xrightarrow{p_f} X) := (P_f \xrightarrow{p_2} PX \xrightarrow{t} X)$ and $Y \xrightarrow{w_f} P_f$ as the unique arrow $Y \to P_f$ such that $(Y \to P_f \xrightarrow{p_1} Y) = 1_Y$ and $(Y \to P_f \xrightarrow{p_2} PX) = (Y \xrightarrow{f} X \xrightarrow{r} P_X)$.*

*In particular, if $f$ is a weak equivalence then such a $p_f$ is also an acyclic fibration (by 5. of Definition 1.3). We will call the couple $(w_f, p_f)$ a* weak-fibre factorization *of $f$.*

**Proposition A.6.** *Let $\mathcal{C}$ be a path category and let $A$ be an object of $\mathcal{C}$. Then the full subcategory $\mathcal{C}(A)$ of $\mathcal{C}/A$ whose objects are the fibrations of $\mathcal{C}$ of target $A$ is a path category as well if: the class of fibrations is the class of the arrows $(X \to A) \xrightarrow{f} (Y \to A)$ such that $X \xrightarrow{f} Y$ is a fibration of $\mathcal{C}$; the class of weak equivalences is the class of the arrows $(X \to A) \xrightarrow{f} (Y \to A)$ such that $X \xrightarrow{f} Y$ is a weak equivalence of $\mathcal{C}$. Moreover, whenever $A \xrightarrow{g} B$ is an arrow of $\mathcal{C}$, then (by 2. of Definition 1.3) the pullback functor $\mathcal{C}/B \xrightarrow{g^*} \mathcal{C}/A$ restricts to a functor $\mathcal{C}(B) \xrightarrow{g^*} \mathcal{C}(A)$ and the latter preserves both the fibrations and the weak equivalences.*

Let $\mathcal{C}$ be a path category and let $X \xrightarrow{f} A$ and $Y \xrightarrow{g} A$ be arrows of $\mathcal{C}$. Then there exists the pullback:

$$\begin{tikzcd} X\, {}^h{\times}_A Y \arrow[r] \arrow[d, "\langle p_1, p_2 \rangle"'] & PA \arrow[d, "\langle s,t \rangle"] \\ X \times Y \arrow[r, "f \times g"] & A \times A \end{tikzcd}$$

of the arrow $\langle s, t\rangle$ along the arrow $f \times g$ (by 2. of Definition 1.3). Hence $fp_1 \simeq gp_2$, that is, the following square (♠):

$$\begin{tikzcd} X\, {}^h{\times}_A Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

commutes *up to homotopy*. We can give the following:

**Definition A.7.** Let $\mathcal{C}$ be a path category and let $X \xrightarrow{f} A$ and $Y \xrightarrow{g} A$ be arrows of $\mathcal{C}$. We say that a diagram:

$$\begin{tikzcd} C \arrow[r, "q_2"] \arrow[d, "q_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

commuting up to homotopy is a *homotopy pullback square* if there is a homotopy equivalence (that is, a weak equivalence) $C \to X\, {}^h{\times}_A Y$ such that $(C \to X\, {}^h{\times}_A Y \xrightarrow{p_1} X) = (C \xrightarrow{q_1} X)$ and $(C \to X\, {}^h{\times}_A Y \xrightarrow{p_2} X) = (C \xrightarrow{q_2} X)$.

Observe that every pair of arrows with the same target has a homotopy pullback since the square ♠ is always a homotopy pullback. Moreover one can prove that the homotopy pullback of a homotopy equivalence along any arrow is a homotopy equivalence as well and that, if such a diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

commutes up to homotopy and the right square is a homotopy pullback, then it is a homotopy pullback if and only if the left square is a homotopy pullback.

**Definition A.8.** Suppose that $g$ and $g'$ are parallel arrows $Y \to X$ of a given path category $\mathcal{C}$ and suppose that there is a fibration $X \to A$ such that $(Y \xrightarrow{g} X \to A) = (Y \xrightarrow{g'} X \to A)$. Let $(P(X \to A) = (P_A(X) \to A), r, \langle s, t\rangle)$ be a triple in the path category $\mathcal{C}(A)$ satisfying 6. of Definition 1.0.1. for the object $X \to A$ of $\mathcal{C}(A)$ and let us assume that there is an arrow $Y \xrightarrow{h} P_A(X)$ such that $\langle g, g'\rangle = \langle s, t\rangle h$ in $\mathcal{C}$ (observe that, if $Y \xrightarrow{g} X \to A$ is a fibration, then this means that $g \simeq g'$ in $\mathcal{C}(A)$ and viceversa). Then we say that $g$ and $g'$ are *fibrewise homotopic (w.r.t. $A$)* and we write $g \simeq_A g'$.

Suppose that $Y \xrightarrow{f} X$ is a fibration of $\mathcal{C}$ with weak-fibre factorization $(Y \xrightarrow{f} X) = (Y \xrightarrow{w_f} P_f \xrightarrow{p_f} X)$ as in Proposition A.5 (observe that $w_f$ is an arrow $(Y \xrightarrow{f} X) \to (P_f \xrightarrow{p_f} X)$ in $\mathcal{C}(X)$). Let $\Gamma$ be an arrow $P_f \to Y$ such that $(P_f \xrightarrow{\Gamma} Y \xrightarrow{f} X) = p_f$ and $(Y \xrightarrow{w_f} P_f \xrightarrow{\Gamma} X) \simeq_X 1_Y$ (the latter makes sense, since $f(\Gamma w_f) = f(1_Y)$ and $f$ is a fibration). Then we say that $\Gamma$ is a *transport structure on $f$*.

Let $PY$ and $PX$ be path objects of $Y$ and $X$ respectively with the structure verifying 6. of Definition 1.3. Let $Pf$ be a fibration $PY \to PX$ commuting with this structure and let $\nabla$ be an arrow $P_f \to PY$ such that $(P_f \xrightarrow{\nabla} PY \xrightarrow{Pf} PX) = (P_f \xrightarrow{p_2} PX)$ and $(P_f \xrightarrow{\nabla} PY \xrightarrow{s} Y) = (P_f \xrightarrow{p_1} Y)$ (see Proposition A.5). Then we say that $(Pf, \nabla)$ is a *connection on $f$*.

As mentioned in the Introduction, the transport structure of a fibration is the categorical counterpart to the concept of *transport* in Homotopy Type Theory. One can prove that:

**Theorem A.9.** *If $Y \xrightarrow{f} X$ is a fibration of a given path category $\mathcal{C}$, then there is a transport structure $P_f \xrightarrow{\Gamma} Y$ on $f$ and any two transport structures on $f$ are fibrewise homotopic over $X$. Moreover there is a connection $(PY \xrightarrow{Pf} PX, P_f \xrightarrow{\nabla} PY)$ on $f$ such that $(P_f \xrightarrow{\nabla} PY \xrightarrow{t} Y) = (P_f \xrightarrow{\Gamma} Y)$.*

*In other words, if $Y \xrightarrow{f} X$ is a fibration of $\mathcal{C}$, then there is a triple:*

$$(P_f \xrightarrow{\Gamma} Y,\ PY \xrightarrow{Pf} PX,\ P_f \xrightarrow{\nabla} PY)$$

*such that $Pf$ is a fibration, commutes with the structure of $PY$ and $PX$ and the following:*

$$(P_f \xrightarrow{\Gamma} Y \xrightarrow{f} X) = (P_f \xrightarrow{p_f} X)$$
$$(Y \xrightarrow{w_f} P_f \xrightarrow{\Gamma} Y) \simeq_X (Y \xrightarrow{1_Y} Y)$$
$$(P_f \xrightarrow{\nabla} PY \xrightarrow{Pf} PX) = (P_f \xrightarrow{p_2} PX)$$
$$(P_f \xrightarrow{\nabla} PY \xrightarrow{s} Y) = (P_f \xrightarrow{p_1} Y)$$
$$(P_f \xrightarrow{\nabla} PY \xrightarrow{t} Y) = (P_f \xrightarrow{\Gamma} Y)$$

*are satisfied, where (as in Proposition A.5) the following square:*

$$\begin{tikzcd} P_f \arrow[r, "p_2"] \arrow[d, "p_1"'] & PX \arrow[d, "s"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

*is a pullback and:*

$$(P_f \xrightarrow{p_f} X) := (P_f \xrightarrow{p_2} PX \xrightarrow{t} X)$$
$$(Y \xrightarrow{w_f} P_f) := \langle Y \xrightarrow{1_Y} Y, Y \xrightarrow{f} X \xrightarrow{r} P_X \rangle.$$

**Remark A.10.** Whenever $Y \xrightarrow{f} X$ is a fibration of $\mathcal{C}$, $(PX, r, \langle s, t\rangle)$ is a path object of $X$ and $\Gamma\colon P_f \to Y$ is a transport structure of $f$, then a couple $(PY \xrightarrow{Pf} PX, P_f \xrightarrow{\nabla} PY)$ verifying the thesis of Theorem A.9 can be obtained as follows.

As $f$ is a fibration, it is the case that $Y \xrightarrow{f} X$ is an object of $\mathcal{C}(X)$, a path category, and then it admits a path object $(P(Y \xrightarrow{f} X) = (P_X Y \to X), r_X, \langle s_X, t_X\rangle)$ in $\mathcal{C}(X)$. If the following square:

$$\begin{tikzcd} P_f \times_Y P_X Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_X Y \arrow[d, "s_X"] \\ P_f \arrow[r, "\Gamma"] & Y \end{tikzcd}$$

is a pullback, let $PY := P_f \times_Y P_X Y$ and let:

$$(Y \xrightarrow{r} PY) := \langle w_f, r_X \rangle \text{ [as } \Gamma w_f = 1_Y = s_X r_X]$$
$$(PY \xrightarrow{s} Y) := (P_f \times_Y P_X Y \xrightarrow{\pi_1} P_f \xrightarrow{p_1} Y)$$
$$(PY \xrightarrow{t} Y) := (P_f \times_Y P_X Y \xrightarrow{\pi_2} P_X Y \xrightarrow{t_X} Y)$$

then $(PY, r, \langle s, t\rangle)$ is a path object. Moreover, let:

$$(PY \xrightarrow{Pf} PX) := (P_f \times_Y P_X Y \xrightarrow{\pi_1} P_f \xrightarrow{p_2} PX)$$
$$(P_f \xrightarrow{\nabla} PY) := \langle 1_{P_f}, (P_f \xrightarrow{\Gamma} Y \xrightarrow{r_X} PY)\rangle \text{ [as } \Gamma 1_{P_f} = s_X(r_X \Gamma)]$$

Then the so-defined couple $(P_f, \nabla)$ does the job.

Finally, we state without proof the following:

**Theorem A.11.** *Let $\mathcal{C}$ be a path category and let $Y \xrightarrow{p} X$ be a fibration. Let $Z \xrightarrow{f} Y$ and let $Z \xrightarrow{g} X$ be arrows of $\mathcal{C}$ such that $(Z \xrightarrow{f} Y \xrightarrow{p} X) \simeq (Z \xrightarrow{g} X)$. Then there is an arrow $Z \xrightarrow{f'} Y$ of $\mathcal{C}$ such that $f' \simeq f$ and $(Z \xrightarrow{f'} Y \xrightarrow{p} X) = (Z \xrightarrow{g} X)$.*

*In particular, if $h$ is a homotopy $pf \simeq g$, then one can take $f' = \Gamma_p\langle f, h\rangle$.*

and the following fundamental:

**Theorem A.12.** *Let $\mathcal{C}$ be a path category and let us assume that the following square:*

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "f"'] & C \arrow[d, "p"] \\ B \arrow[r, "n"] & D \end{tikzcd}$$

*commutes. Moreover let us assume that $p$ is a a fibration and $f$ is a weak equivalence. Then there is an arrow $B \xrightarrow{l} C$ such that $n = pl$ and $lf \simeq_D m$ and it is unique up to fibrewise homotopy w.r.t. $D$.*

The latter proves the uniqueness up to homotopy equivalence of the weak equivalence-fibration factorization in a path category:

**Theorem A.13.** *Let $\mathcal{C}$ be a path category and let us assume that the following square:*

$$\begin{tikzcd} Y \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d, "p"] \\ B \arrow[r, "q"] & X \end{tikzcd}$$

*commutes. Moreover let us assume that the arrows $a$ and $b$ are weak equivalences and that $p$ and $q$ are fibrations. Then there is a homotopy equivalence $A \xrightarrow{f} B$ with* homotopy inverse $B \xrightarrow{g} A$ *(that is, $[g][f] = [1_A]$ and $[f][g] = [1_B]$ in* $\mathrm{Ho}(\mathcal{C})$*) such that $gf \simeq_X 1_A$, $fg \simeq_X 1_B$, $fa \simeq_X b$, $gb \simeq_X a$,$qf = p$ and $pg = q$.*

## A.2 Some Needed Lemmas and Remarks

This section contains basic technical results that we use during the chapters.

**Lemma A.14.** *Let $\mathcal{C}$ be a category and let $A \xrightarrow{f} B$ be a regular epimorphism of $\mathcal{C}$, that is, in $\mathcal{C}$ there is a parallel pair of codomain $A$ whose coequalizer exists and is $f$. Moreover, let us assume that $f$ has a kernel pair. Then $f$ is the coequalizer of its own kernel pair.*

*Proof.* Let $\alpha$ and $\beta$ be arrows $C \to A$ such that $f$ is the coequalizer of $\alpha$ and $\beta$. Let $\gamma$ and $\delta$ be arrows $D \to A$ such that the following square:

$$\begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "\gamma"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback. Then there is unique an arrow $C \xrightarrow{x} D$ such that $\gamma x = \alpha$ and $\delta x = \beta$. Therefore, whenever $g$ is an arrow $A \to E$ coequalizing $\gamma$ and $\delta$, it also coequalizes $\alpha$ and $\beta$. Hence there is an arrow $B \xrightarrow{h} E$ such that $hf = g$. We conclude that $f$ is a coequalizer of $\gamma$ and $\delta$. Q.E.D.

**Lemma A.15.** *Let $\mathcal{C}$ be a category and let $\alpha$ and $\beta$ be parallel arrows $R \to A$ of $\mathcal{C}$ such that the couple $(\alpha, \beta)$ is a kernel pair (of some arrow of $\mathcal{C}$). Moreover let us assume that the couple $(\alpha, \beta)$ has a coequalizer. Then $(\alpha, \beta)$ is the kernel pair of its own coequalizer.*

*Proof.* Let $f \colon A \to B$ be an arrow of $\mathcal{C}$ such that the following square:

$$\begin{tikzcd} R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback and let $A \xrightarrow{q} C$ be a coequalizer of $\alpha$ and $\beta$. Since $f$ coequalizes $\alpha$ and $\beta$, there is unique an arrow $C \xrightarrow{x} B$ such that $xq = f$. Now, let $\gamma$ and $\delta$ be parallel arrows $S \to A$ such that $q\gamma = q\delta$. Then $f\gamma = f\delta$ and therefore there is unique an arrow $S \xrightarrow{h} R$ such that $\alpha h = \gamma$ and $\beta h = \delta$. We conclude that the couple $(\alpha, \beta)$ is a kernel pair of $q$. Q.E.D.

**Lemma A.16.** *An isomorphism of a given category is nothing but a monomorphism with a section. Dually, an isomorphism of a given category is nothing but a epimorphism with a retraction.*

*Proof.* Let $A \xrightarrow{m} B$ be a monomorphism and let $s$ be a section of $m$. Then we observe that $msm = 1_B m = m = m1_A$, hence $sm = 1_A$. Since $ms = 1_B$ we are done. The viceversa is clear, as every isomorphism is in particular a monomorphism and its inverse is in particular a section. Q.E.D.

**Remark A.17.** Let $\mathcal{C}$ be a category with terminal object 1 and let $S$ and $T$ be objects of $\mathcal{C}$. Let us assume that, for every object $I$ in $\mathcal{C}$, there is a map $\alpha_I \colon \mathcal{C}(I, S) \to \mathcal{C}(I, T)$ such that, for every arrow $J \to I$ of $\mathcal{C}$, the following diagram:

$$\begin{tikzcd} \mathcal{C}(I,S) \arrow[r, "\alpha_I"] \arrow[d, "-\circ(J\to I)"'] & \mathcal{C}(I,T) \arrow[d, "-\circ(J\to I)"] \\ \mathcal{C}(J,S) \arrow[r, "\alpha_J"] & \mathcal{C}(J,T) \end{tikzcd}$$

commutes. In other words, let us assume that $\alpha = \{\alpha_I\}_{I \text{ in } \mathcal{C}}$ is a natural transformation $\mathcal{C}(-, S) \to \mathcal{C}(-, T)$. Then, by Yoneda's Lemma, there is unique an arrow $S \xrightarrow{\overline{\alpha}} T$ such that, for every object $I$ of $\mathcal{C}$ and every $x \in \mathcal{C}(I, S)$, it is the case that $\alpha_I(x) = (I \xrightarrow{x} S \xrightarrow{\overline{\alpha}} T)$.

In this case, we define the arrow $S \xrightarrow{\overline{\alpha}} T$ as *the unique arrow $S \xrightarrow{f} T$ that in a better world would be such that $f(x) = \alpha_1(x)$ for every $x \in S$.*

**Lemma A.18.** *Let $g$ and $g'$ be parallel arrows $Y \to X$ of a path category $\mathcal{C}$ such that there is a fibration $X \xrightarrow{\alpha} A$ such that $\alpha g = \alpha g'$ and $g \simeq_A g'$. Then $g \simeq g'$.*

*Proof.* Let us consider the pullback:

$$\begin{tikzcd} X \times_A X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "\alpha"] \\ X \arrow[r, "\alpha"'] & A \end{tikzcd}$$

whose unique arrow $X \times_I X \xrightarrow{\alpha\times\alpha} A$ is a fibration. Let us consider a path object $(P_A(X) \to A, r_A, \langle s_A, t_A\rangle)$ of $\alpha$. Then $s_A$ and $t_A$ are arrows $P_A(X) \to X$ such that $\alpha s_A = \alpha t_A$, and $\langle s_A, t_A\rangle$ is the induced arrow $P_A(X) \to X \times_A X$. Let us consider the arrow $X \times_A X \xrightarrow{p=\langle p_1, p_2\rangle} X \times X$ and let us observe that $\pi_1 p\langle s_A, t_A\rangle r_A = p_1\langle s_A, t_A\rangle r = s_A r = 1_X$ and analogously $\pi_2 p\langle s_A, t_A\rangle r_A = 1_X$ (here $\pi_1$ and $\pi_2$ are the projections $X \times X \to X$ of the product $X \times X$). Hence $p\langle s_A, t_A\rangle r_A = \delta_X$. In particular, the following diagram:

$$\begin{tikzcd} X \arrow[rr, "r"] \arrow[d, "r_A"'] & & PX \arrow[d, "\langle s,t\rangle"] \\ P_A X \arrow[r, "\langle s_A, t_A\rangle"] & X \times_A X \arrow[r, "p"] & X \times X \end{tikzcd}$$

commutes, being $(PX, r, \langle s, t\rangle)$ a path object over $X$. Being $r_A$ a weak equivalence and being $\langle s, t\rangle$ a fibration, by Theorem A.12 there is an arrow $P_A X \xrightarrow{l} PX$ such that the square:

$$\begin{tikzcd} P_AX \arrow[r, "l"] \arrow[d, "\langle s_A,t_A\rangle"'] & PX \arrow[d, "\langle s,t\rangle"] \\ X\times_A X \arrow[r, "p"] & X\times X \end{tikzcd}$$

commutes.

Finally, if $Y \xrightarrow{h} P_AX$ is a fibrewise homotopy $g \simeq_A g'$, then:

$$\begin{aligned} \langle g, g'\rangle_{X\times X} &= p\langle g, g'\rangle_{X\times_A X} \\ &= p\langle s_A, t_A\rangle h \\ &= \langle s,t\rangle(lh) \end{aligned}$$

and we are done. Q.E.D.

**Lemma A.19.** *Let $\mathcal{C}$ be a path category and let $g, h$ be arrows $Y \to X$ such that $fg = fh$ for some $X \xrightarrow{f} Z$. Moreover, let us assume that there is an equivalence $Y' \xrightarrow{w} Y$ such that $gw \simeq_Z hw$. Then $g \simeq_Z h$.*

*Proof.* If $k$ is a homotopy $gw \simeq_Z hw$, then the diagram:

$$\begin{tikzcd} Y' \arrow[r, "h"] \arrow[d, "w"'] & P_ZX \arrow[d, "\langle s_Z,t_Z\rangle"] \\ Y \arrow[r, "\langle g,h\rangle"] & X\times X \end{tikzcd}$$

commutes. Being $\langle s_Z, t_Z\rangle$ a fibration and being $w$ a weak equivalence, by Theorem A.12 there is an arrow $Y \to P_ZX$ making the lower triangle commute, that is, a homotopy $g \simeq_Z h$. Q.E.D.

**Lemma A.20.** *Let $\mathcal{C}$ be a path category and let $Y \xrightarrow{f} X$ be a fibration. Let $g$ be a section of $f$. Then there is a homotopy $\Gamma_f\langle gs, 1_{PX}\rangle \simeq_X gt$.*

*Proof.* We use the construction provided by Theorem A.9 and Remark A.10. Observe that $f\Gamma_f\langle gs, 1_{PX}\rangle = p_f\langle gs, 1_{PX}\rangle = tp_2\langle gs, 1_{PX}\rangle = t = fgt$. Moreover $\Gamma_f\langle gs, 1_{PX}\rangle r = \Gamma_f\langle g, r\rangle = \Gamma_f\langle g, rfg\rangle = \Gamma_f\langle 1_Y, rf\rangle g = \Gamma_f w_f g \simeq_X 1_Y g = g = gtr$. By Lemma A.19 and being $r$ a weak equivalence, it is the case that $\Gamma_f\langle gs, 1_{PX}\rangle \simeq_X gt$. Q.E.D.

Let $\mathcal{C}$ be a path category and let $Y \xrightarrow{f} X$ be a fibration. Let us consider the constructions of Remark A.10 and the pullbacks:

$$\begin{tikzcd} P_XY\times_X PX \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & PX \arrow[d, "s"] \\ P_XY \arrow[r, "o"] & X \end{tikzcd} \qquad \begin{tikzcd} P_f \arrow[r, "p_2"] \arrow[d, "p_1"'] & PX \arrow[d, "s"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

and observe that $s\beta_2 = o\beta_1 = fs_X\beta_1$ and that $s\beta_2 = o\beta_1 = ft_X\beta_1$. Hence, with respect to the right pullback, we can consider the arrows $\langle s_X\beta_1, \beta_2\rangle$ and $\langle t_X\beta_1, \beta_2\rangle$. Then the following holds:

**Theorem A.21.** *Let $\mathcal{C}$ be a path category and let $Y \xrightarrow{f} X$ be a fibration. Then there are homotopies*

$$\Gamma_f\langle s_X\beta_1, \beta_2\rangle \simeq_X \Gamma_f\langle t_X\beta_1, \beta_2\rangle \text{ and } \Gamma_f\langle \Gamma_f\langle p_1, p_2\rangle, \sigma p_2\rangle \simeq_X p_1,$$

*being $\sigma$ the result of the application of Theorem A.12 to the diagram:*

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "r"'] & PX \arrow[d, "\langle s,t\rangle"] \\ PX \arrow[r, "\langle t,s\rangle"] & X \times X. \end{tikzcd}$$

*Proof.* See [12] - Corollary 3.2.3 and [13] - Proposition 2.12. Q.E.D.

The following proposition clarifies why we use the terminology *homotopy pullback* (see Definition A.7): it is a weak pullback up to homotopy.

**Proposition A.22.** *Let $\mathcal{C}$ be a path category. Then the quotient functor $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ sends the homotopy pullbacks to weak pullbacks. In particular $\mathrm{Ho}(\mathcal{C})$ is weakly finitely complete.*

*Proof.* Let $X \xrightarrow{f} A$ and $Y \xrightarrow{g} A$ be arrows of $\mathcal{C}$ and let us consider the square:

$$\begin{tikzcd} X\,{}^h{\times}_A Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

of Definition A.7. Let $\alpha$ and $\beta$ be arrows $Z \to X$ and $Z \to Y$ respectively, such that $f\alpha \simeq g\beta$. Then there is an arrow $Z \xrightarrow{h} PA$ such that $(f \times g)\langle\alpha, \beta\rangle = \langle s, t\rangle h$. Then (see Definition A.7) there is an arrow $Z \xrightarrow{h'} X\,{}^h{\times}_A Y$ such that $\langle p_1, p_2\rangle h' = \langle\alpha, \beta\rangle$ and we are done. Q.E.D.

**Lemma A.23.** *Let $\mathcal{C}$ be a path category and let $Y \xrightarrow{u} X$ be a section of a weak equivalence $X \xrightarrow{l} Y$. Then there is a homotopy $h\colon ul \simeq 1_X$ such that $hu \simeq_{X\times X} ru$, being $(PX, r, \langle s, t\rangle)$ a path object over $X$.*

*Proof.* Since $lu = 1_Y$, it is the case that $u$ is a weak equivalence. Since the diagram:

$$\begin{tikzcd} Y \arrow[r, "ru"] \arrow[d, "u"'] & PX \arrow[d, "\langle s,t\rangle"] \\ X \arrow[r, "\langle ul,1_X\rangle"] & X \times X \end{tikzcd}$$

commutes, the arrow $\langle s, t\rangle$ is a fibration and the arrow $u$ is a weak equivalence, by Theorem A.12, there is an arrow $h\colon X \to PX$ such that $\langle s, t\rangle h = \langle ul, 1_X\rangle$ and that $hu \simeq_{X\times X} ru$.
Q.E.D.

## References


[1] Peter Johnstone.
*Topos Theory*, 1977.
Academic Press, Inc., New York, 1st edition.

[2] Aurelio Carboni & R. Celia Magno.
*The Free Exact Category on a Left Exact One*, 1982.
Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics, 33(3), 295-301. doi:10.1017/S1446788700018735.

[3] Saunders Mac Lane & Ieke Moerdijk.
*Sheaves in Geometry and Logic*, 1994.
Springer-Verlag, New York, 1st edition.

[4] André Joyal & Ieke Moerdijk.
*Algebraic Set Theory*, 1995.
Cambridge University Press.

[5] Aurelio Carboni.
*Some Free Constructions in Realizability and Proof Theory*, 1995.
Journal of Pure and Applied Algebra, Volume 103, Issue 2, Pages 117-148.

[6] Aurelio Carboni & Enrico M. Vitale.
*Regular and Exact Completions*, 1998.
Journal of Pure and Applied Algebra, Volume 125, Issues 1–3, Pages 79-116.

[7] Saunders Mac Lane.
*Categories for the working mathematician*, 1998.
Springer-Verlag, New York, 2nd edition.

[8] Matías Menni.
*Exact Completions and Toposes*, 2000.
PhD Thesis, University of Edinburgh.

[9] Benno van den Berg & Ieke Moerdijk.
*Exact Completion of Path Categories and Algebraic Set Theory*, 2016.
Journal of Pure and Applied Algebra, Volume 222, Pages 3137-3181.

[10] Emily Riehl.
*Category Theory in Context*, 2016.
Dover Publications, Inc., Mineola, New York, 1st edition.

[11] Benno van den Berg.
*Path Categories and Propositional Identity Types*, 2016.
Transactions on Computational Logic. arXiv: 1604.06001 [math.CT].

[12] Menno de Boer.
*The Gluing Construction for Path Categories*, 2018.
Master's Thesis, Utrecht.

[13] Benno van den Berg.
*Univalent Polymorphisms*, 2018.
ArXiv e-prints. arXiv: 1803.10113 [math.CT].

Ringrazio ogni persona con cui abbia avuto un contatto sincero durante questi anni autentici di studente di matematica.