Elements of Topological Quantum Field Theory

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Introduction

This essay is mostly about four notions: the notion of *(symmetric) monoidal structure* over a category (together with the related notion of (symmetric) monoidal property of a functor), the notion of *cobordism* (and the consequent notion of category of cobordisms), the notion of *topological quantum field theory* (that provides an instance of symmetric monoidal functor) and the notion of *Frobenius algebra*. In our formulation a monoidal category is nothing but a *monoid-object* in the category of the categories and the functors between them. We briefly discuss this notion in Section 1. In this section we also present other basic concepts, related to the ones of monoidal category and monoidal functor, that we will need in the other sections to talk about the categories of cobordisms and in order to present the definition of n-dimensional TQFT w.r.t. a given field and for a given positive integer n.

In Section 2 we discuss the notion of n-dimensional cobordism (for a given positive integer n) and we show how the equivalence classes of n-dimensional cobordisms (modulo a precise equivalence relation) can be seen as the arrows of a category (called the *n-dimensional* *cobordism category*) whose objects are the $(n-1)$-dimensional oriented compact smooth manifolds without boundary. After that, we enumerate a list of basic facts about the cobordism categories and we show how they are endowed with a symmetric monoidal structure. Finally we define the TQFTs of a given dimension n (for a given positive integer n) w.r.t. a given field $\mathbb{K}$ as the $\mathbb{K}$-*linear representations* of the n-dimensional cobordism category.

The aim of this essay is to describe the bidimensional TQFTs. In order to do this, in Section 3 we present and prove a *monoidal presentation* of the bidimensional cobordism category 2Cob. A monoidal presentation is made of a collection of arrows of 2Cob together with a collection of equalities only involving the symmetric monoidal categorical structure of 2Cob, in such a way that the whole information about the symmetric monoidal categorical structure of 2Cob can be deduced by these two collections. In fact we require the collection of arrows to be such that every other arrow can be obtained by applying the symmetric monoidal categorical structure of 2Cob finitely many times on this collection, and the collection of equalities to be such that any other equality only involving the symmetric monoidal categorical structure can be deduced by them. As we are interested in the description of the bidimensional TQFTs and as these are symmetric monoidal functors *from* 2Cob and therefore preserve the isomorphisms, it is enough to determine a presentation of a skeleton of 2Cob.

As every arrow of such a skeleton is obtained by its categorical and symmetric monoidal structures and by the arrows of the determined presentation (and as a symmetric monoidal functor, like for instance a TQFT, preserves both the categorical and the symmetric monoidal structures), the behaviour of a bidimensional TQFT is determined by the values that it assumes on these arrows. Moreover, the equalities of the presentation provide a set of relations between these values. In Section 4 we show that these values in $\mathbb{K}$-Vect assumed by a given TQFT (restricted to the given skeleton of 2Cob) together with this set of relations can be equivalently summarised in a particular structure made of a $\mathbb{K}$-linear space together with four $\mathbb{K}$-linear maps satisfying certain properties only involving the symmetric monoidal and categorical structures of $\mathbb{K}$-Vect. A subsection of Section 4 is devoted to prove that such a structure is a commutative Frobenius $\mathbb{K}$-algebra and, viceversa, that *every* commutative Frobenius $\mathbb{K}$-algebra can be obtained from a bidimensional TQFT by applying this procedure. Moreover, in the ending part of Section 4 we reinforce this result by proving that it is the case that the category of the bidimensional TQFTs (w.r.t. the field $\mathbb{K}$) (together with a natural choice of arrows between them) is equivalent to the category of the commutative Frobenius $\mathbb{K}$-algebras (again, together with a natural choice of arrows between them). Finally, we observe that one may generalise this result to any symmetric monoidal category in place of $\mathbb{K}$-Vect.

The Appendix is made of four parts: the first one presents the formal definition of the notion of *classic model* of a given first-order theory; the second one briefly illustrates some categorical notation widely used during the essay; the third one enumerates the basic category-theoretic notions appearing through the sections; the fourth one briefly explains an uncommon set-theoretic notation that is used in a couple of arguments.

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1 Monoidal Categories

We begin this section by introducing the notion of *monoidal category*, which is fundamental in order to present the formulation of a TQFT that we will be concerned with. In the literature there exist quite different versions of this notion, some of them weaker than the others (one can see [1, 2] to have a complete overview of them). However, the version that we will be concerned with is not the weakest one. Hence we do not need to present the theory of monoidal categories in the highest degree of generality.

Whenever we are given a *first-order language* (that is, a set of *function-symbols*, each of them with a given *arity*, and of *relation-symbols*, again with their arity), we say that a *(classic) first-order structure* for this language is just a set with an auto-function of a given arity for every function-symbol (in the given language) of that arity, and a relation of a given arity for every relation-symbol (in the given language) of that arity. For instance, if the given language is $L := \{f, g, c, R\}$, where f is a binary function-symbol, g is a 1-ary function-symbol, c is a 0-ary function-symbol (that is, a constant-symbol) and R is a binary relation-symbol, then an L-(classic) first-order structure is a set X with the following data: a function $f' \colon X \times X \to X$, a function $g' \colon X \to X$, a constant $c' \colon 1 = \{0\} \to X$ (that is, an element $c' \in X$) and a relation $R' \subseteq X \times X$ (see Appendix **1.**).

Moreover, if T is a *first-order theory* in the given language (that is, a set of first-order formulas -without free variables- only concerning the usual logic symbology, the function-symbols and the relation-symbols of that language), we say that a structure for this language is a *classic model* of T if this structure *verifies* (see Appendix **1.** for a formal definition of this notion) all the properties contained in T. Let us present a couple of instances of this notion:

1. Let $L := \{R\}$, where R is a binary relation-symbol. Let T be the set whose elements are $(\forall x)(xRx)$, $(\forall x \forall y)(xRy \wedge yRx \implies x = y)$ and $(\forall x \forall y \forall z)(xRy \wedge yRz \implies xRz)$. Then T is the so-called *theory of posets*. According to the given definition, an L-structure (X, R') -remind that X is a set and $R' \subseteq X \times X$- is a model of T if and only if it verifies the elements of T, that is, if it is a poset. In other words, the classic models of the theory of posets are precisely the posets.

2. Let $L := \{f, c\}$, where f is a binary function-symbol and c is a 0-ary function-symbol. Let T be the theory of monoids, that is, the elements of T are $(\forall x \forall y \forall z)(f(f(x, y), z) = f(x, f(y, z)))$ and $(\forall x)(f(x, c) = x = f(c, x))$. As before, a set X with a binary function $h \colon X \times X \to X$ and a constant $e \in X$ constitutes a model of T if and only if it verifies these two axioms, that is, if it is a monoid. Again, we saw that the classic models of the theory of monoids are precisely the monoids.

For a more detailed explanation of these concepts, see Appendix **1.**. In its categorical environment, logic arises from the observation that *models* of a given first-order theory may be defined in an arbitrary (sufficiently rich) category, not necessarily Set, the category whose objects are the sets and whose morphisms are the functions between them. The classic models of a given first-order theory turn out to be its *models* (according to the generalised notion of model that we are presenting) in the event that the chosen category is Set. Let us explain this phenomenon through an instance of it.

Let us consider the language $L = \{f, c\}$ of the second example and the theory T of monoids. The properties of T that an L-structure (X, h, e) needs to verify in order to be a monoid can be equivalently formulated by requiring that the following diagrams:

$$
\begin{tikzcd}
X \times X \times X \arrow[r, "h \times 1_X"] \arrow[d, "1_X \times h"'] & X \times X \arrow[d, "h"] \\
X \times X \arrow[r, "h"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times 1 \arrow[d, "1_X \times e"'] & X \arrow[l, "\langle 1_X , ! \rangle"'] \arrow[r, "\langle !, 1_X \rangle"] \arrow[d, "1_X"] & 1 \times X \arrow[d, "e \times 1_X"] \\
X \times X \arrow[r, "h"'] & X & X \times X \arrow[l, "h"]
\end{tikzcd}
$$

commute (if some notation of the objects and the arrows of these diagrams is not clear, see Appedix **2.**). This characterisation constitutes a category-theoretic definition of the notion of monoid, as it only talks about objects of Set (namely the sets 1, X and products of them) and arrows of Set (the functions $h\colon X \times X \to X$ and $e\colon 1 \to X$ and products of them), without mentioning elements of sets. Hence we can use it to express what a *monoid-object* in a category $\mathcal{C}$ with finite products (that is, a *model* of the theory of monoids in the category $\mathcal{C}$) is. It is just an object X of $\mathcal{C}$ together with two morphisms $h\colon X \times X \to X$ and $e\colon 1 \to X$ of $\mathcal{C}$ such that the previous diagrams commute. In particular, it is the case that the monoids are precisely the monoid-objects in Set, that is, the models of the theory of monoids in Set. As we saw that the monoids are exactly the classic models of the theory of monoids, we concude that the classic models of the theory of monoids are precisely the models of the theory of monoids in Set. Actually, this result holds for any first-order theory.

Now we can give the following:

Definition 1.1. A *monoidal category* is just a monoid-object in the category of categories, that is, a model of the theory of monoids in the category of categories.

According to Definition 2.1, a monoidal category is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a a functor $\eta\colon \mathtt{1} \to \mathcal{C}$, being $\mathtt{1}$ the terminal category (for the definition of *terminal object* of a category, see Appendix **2.**), such that the following diagrams:

$$
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes \times 1_{\mathcal{C}}"] \arrow[d, "1_{\mathcal{C}} \times \otimes"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \times \mathtt{1} \arrow[d, "1_{\mathcal{C}} \times \eta"'] & \mathcal{C} \arrow[l, "\langle 1_{\mathcal{C}} , ! \rangle"'] \arrow[r, "\langle !, 1_{\mathcal{C}} \rangle"] \arrow[d, "1_{\mathcal{C}}"] & \mathtt{1} \times \mathcal{C} \arrow[d, "\eta \times 1_{\mathcal{C}}"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C} & \mathcal{C} \times \mathcal{C} \arrow[l, "\otimes"]
\end{tikzcd}
$$

commute, being ! the unique functor $\mathcal{C} \to \mathtt{1}$ (whose existence and whose uniqueness are guaranteed by the terminality of the category $\mathtt{1}$). Actually, the notion of *monoidal category* that we are working with is just a particular case of the one that usually appears in the literature (see [1]). Here a monoidal category is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a functor $\eta\colon \mathtt{1} \to \mathcal{C}$ and three natural isomorphisms (see Appendix **3.**):

$$\alpha\colon (\mathcal{C} \times \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes \times 1_{\mathcal{C}}} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}) \cong (\mathcal{C} \times \mathcal{C} \times \mathcal{C} \xrightarrow{1_{\mathcal{C}} \times \otimes} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C})$$

$$\beta\colon (\mathcal{C} \xrightarrow{\langle 1_{\mathcal{C}}, ! \rangle} \mathcal{C} \times \mathtt{1} \xrightarrow{1_{\mathcal{C}} \times \eta} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}) \cong (\mathcal{C} \xrightarrow{1_{\mathcal{C}}} \mathcal{C})$$

$$\gamma\colon (\mathcal{C} \xrightarrow{\langle !, 1_{\mathcal{C}} \rangle} \mathtt{1} \times \mathcal{C} \xrightarrow{\eta \times 1_{\mathcal{C}}} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}) \cong (\mathcal{C} \xrightarrow{1_{\mathcal{C}}} \mathcal{C})$$

such that the following diagrams:

$$
\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[rr, "\alpha_{(A \otimes B, C, D)}"] \arrow[d, "\otimes \circ (\alpha \times 1_{1_{\mathcal{C}}})_{(A,B,C,D)}"] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{(A,B,C \otimes D)}"'] \\
(A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{(A, B \otimes C, D)}"'] & A \otimes ((B \otimes C) \otimes D) \arrow[r, "\otimes \circ (1_{1_{\mathcal{C}}} \times \alpha)_{(A,B,C,D)}"'] & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}
$$

$$\begin{tikzcd}
A \otimes (\eta(*) \otimes B) \arrow[rr, "\alpha_{(A,\eta(*),B)}"] \arrow[dr, "\otimes\circ(1_{1_{\mathcal{C}}}\times\gamma)_{(A,B)}"'] & & (A \otimes \eta(*)) \otimes B \arrow[dl, "\otimes\circ(\beta\times 1_{1_{\mathcal{C}}})_{(A,B)}"] \\
& A \otimes B &
\end{tikzcd}$$

commute for every choice of the objects A, B, C, D in $\mathcal{C}$, being $*$ the unique object of $\mathtt{1}$.

The natural isomorphism α constitutes a weaker notion of associativity of the operation $\otimes$. Anologously, the natural isomorphisms β and γ should be considered as a weaker way of expressing that $\eta(*)$ is a neutral element w.r.t. the operation $\otimes$. The commutativity of these two diagrams of natural transformations expresses that a priori different applications of the properties represented by α, β and γ are actually the same ones. Of course, if we require that α, β and γ are identities, then the commutativities of the diagrams are vacuously satisfied and we come back to the "stronger" notion of *monoidal category* that we are involved with. Actually, a theorem states that every "weak" monoidal category is equivalent to a "strong" monoidal category (Definition 1.1) through a functor that preserves the monoidal structure (see [2] - Section XI.3; here "strong" monoidal categories verifying Definition 1.1 are called *strict*). This result allows us to only consider strong monoidal categories, that is, to always assume that the isos α, β and γ are identities. This is why we reserve the notation of *monoidal category* for these ones, in line with Definition 1.1.

We conclude this section by introducing some fundamental terminology and presenting some observations regarding it.

Definition 1.2. Let $(\mathcal{C}, \otimes, \eta)$ be a monoidal category. Let us assume that there is a natural isomorphism $\sigma\colon \otimes \to \otimes \circ (\pi_2 \times \pi_1)$, being π_1 and π_2 the usual functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ exhibiting $\mathcal{C} \times \mathcal{C}$ as a product of $\mathcal{C}$ with itself (hence $\pi_2 \times \pi_1$ is the functor sending a morphism $(f, f')\colon (C, C') \to (D, D')$ of $\mathcal{C} \times \mathcal{C}$ to the morphism $(f', f)\colon (C', C) \to (D', D)$). Then we say that $(\mathcal{C}, \otimes, \eta, \sigma)$ is a *symmetric monoidal category.*

Example 1.3 (*Symmetric monoidal structure over* $\mathbb{K}$-Vect)**.** Let $\mathbb{K}$ be a field and let us consider the category $\mathbb{K}$-Vect whose objects are the $\mathbb{K}$-linear spaces and whose arrows are the $\mathbb{K}$-linear maps between them. Let $\otimes$ be the functor $\mathbb{K}\text{-Vect} \times \mathbb{K}\text{-Vect} \to \mathbb{K}\text{-Vect}$ that sends a morphism $(f, f')\colon (V, V') \to (W, W')$ of $\mathbb{K}\text{-Vect} \times \mathbb{K}\text{-Vect}$ to $f \otimes f'\colon V \otimes V' \to W \otimes W'$. Let $\mathbb{K}$ be the functor $\mathtt{1} \to \mathbb{K}$-Vect such that $* \mapsto \mathbb{K}$ (here $\mathbb{K}$ is considered as a $\mathbb{K}$-linear space over itself). Let $\sigma\colon \otimes \to \otimes \circ (\pi_2 \times \pi_1)$ be the natural isomorphism such that, for every choice of $\mathbb{K}$-linear spaces V and W, it is the case that $\sigma_{(V,W)}$ is the unique linear isomorphism $V \otimes W \to W \otimes V$ such that $\sigma_{(V,W)}(v \otimes w) = w \otimes v$ for every $v \in V$ and every $w \in W$. Then $(\mathbb{K}\text{-Vect}, \otimes, \mathbb{K}, \sigma)$ is a symmetric monoidal category.

Remark 1.4. In Example 1.3 we are implicitly meaning that $(U \otimes V) \otimes W = U \otimes (V \otimes W)$ and that $U \otimes \mathbb{K} = \mathbb{K} = \mathbb{K} \otimes U$ for every choice of objects U, V and W in $\mathbb{K}$-Vect. Actually one could say that there are just isomorphisms $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ and $U \otimes \mathbb{K} \cong \mathbb{K} \cong \mathbb{K} \otimes U$ natural in U, V and W (it depends on how they intend a linear space to be), so that $\mathbb{K}$-Vect is a monoidal category in the weaker sense. Anyway, as these isomorphisms are natural and unique, we may identify the left and the right member of everyone of them (this corresponds to considering a *quotient* of the given category) and come back to the official version of $\mathbb{K}$-Vect of Example 1.3. This is precisely an instance of the main argument needed to prove that every "weak" monoidal category is equivalent to a "strong" monoidal category.

Definition 1.5. Let $(\mathcal{C}, \otimes, \eta)$ and $(\mathcal{D}, \otimes', \xi)$ be monoidal categories and let F be a functor $\mathcal{C} \to \mathcal{D}$. Let us assume that there are a natural transformation $F_2\colon \otimes' \circ (F \times F) \to F \circ \otimes$

and a natural transformation $F_0\colon \xi \to F \circ \eta$. Then we say that the triple (F, F_2, F_0) is a *weak monoidal functor* from $(\mathcal{C}, \otimes, \eta)$ to $(\mathcal{D}, \otimes', \xi)$.

If the natural transformations F_2 and F_0 are identities, that is, $\otimes' \circ (F \times F) = F \circ \otimes$ and $\xi = F \circ \eta$, then we say that F is a *monoidal functor* from $(\mathcal{C}, \otimes, \eta)$ to $(\mathcal{D}, \otimes', \xi)$.

Let us assume that σ and τ are natural isomorphisms $\otimes \to \otimes \circ (\pi_2 \times \pi_1)$ and $\otimes' \to \otimes' \circ (\pi_2 \times \pi_1)$. In other words, let us assume that $(\mathcal{C}, \otimes, \eta, \sigma)$ and $(\mathcal{D}, \otimes', \xi, \tau)$ are symmetric monoidal categories. Let us assume that F is a monoidal functor between $(\mathcal{C}, \otimes, \eta)$ and $(\mathcal{D}, \otimes', \xi)$. Moreover, let us assume that $F \circ \sigma = \tau \circ F$, that is, for every choice of object C and C' of $\mathcal{C}$, it is the case that $F\sigma_{(C,C')} = \tau_{(FC,FC')}$. Then we say that F is a *symmetric monoidal functor* from $(\mathcal{C}, \otimes, \eta, \sigma)$ to $(\mathcal{D}, \otimes', \xi, \tau)$.

Definition 1.6. Let $(\mathcal{C}, \otimes, \eta)$ and $(\mathcal{D}, \otimes', \xi)$ be monoidal categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be monoidal functors between them. Let α be a natural transformation $F \to G$ such that $(F\eta(*) \xrightarrow{\alpha_{\eta(*)}} G\eta(*)) = (\xi(*) \xrightarrow{1_{\xi(*)}} \xi(*))$ and $(F(C \otimes C') \xrightarrow{\alpha_{C \otimes C'}} G(C \otimes C')) = (FC \otimes' FC' \xrightarrow{\alpha_C \otimes' \alpha_{C'}} GC \otimes' GC')$ for every choice of objects C and C' of $\mathcal{C}$. Then we say that α is a *monoidal natural transformation* from F to G.

Remark 1.7. As usual there is a category, denoted as MonCat [SymMonCat respectively], whose objects are the monoidal categories [symmetric monoidal cateogories respectively] and whose arrows are the monoidal functors [symmetric monoidal functors respectively] between them. Moreover, whenever $(\mathcal{C}, \otimes, \eta[, \sigma])$ and $(\mathcal{D}, \otimes', \xi[, \tau])$ are monoidal categories [symmetric monoidal cateogories respectively] it is the case that the class:

$$\text{MonCat}((\mathcal{C}, \otimes, \eta), (\mathcal{D}, \otimes', \xi))\ [\text{SymMonCat}((\mathcal{C}, \otimes, \eta, \sigma), (\mathcal{D}, \otimes', \xi, \tau)) \text{ respectively}]$$

of the arrows $(\mathcal{C}, \otimes, \eta[, \sigma]) \to (\mathcal{D}, \otimes', \xi[, \tau])$, that is, the monoidal functors [symmetric monoidal functors respectively] from $(\mathcal{C}, \otimes, \eta[, \sigma])$ to $(\mathcal{D}, \otimes', \xi[, \tau])$ is a category as well: its arrows are the monoidal natural transformations between them. Therefore MonCat [SymMonCat] is endowed with a structure of 2-category.

Let $\mathbb{K}$ be a field and let V be a $\mathbb{K}$-linear space. We recall that a $\mathbb{K}$-linear representation of a group G on V is just a group homomorphism $f\colon G \to \mathrm{Aut}(V)$, that is, a monoid homomorphism $f\colon G \to \mathbb{K}\text{-Vect}(V, V)$, since a monoid homomorphism sends invertible elements to invertible elements. Analogously, a $\mathbb{K}$-linear representation of a monoid M on V is a homomorphism $f\colon M \to \mathbb{K}(V, V)$.

We recall that a monoid M is a small category with one object $\bullet$ (and viceversa). The arrows $\bullet \to \bullet$ are the elements of M and the composition is the associative operation of M. Hence the identity arrow $\bullet \to \bullet$ is the neutral element of M. The monoid M is a group precisely when all the arrows of the corresponding categorical structure are isomorphisms, that is, when all the elements of M are invertible (as usual). Now, with this formulation a $\mathbb{K}$-linear representation of M is just a functor $f\colon M \to \mathbb{K}\text{-Vect}$. Indeed, if f is a functor $M \to \mathbb{K}\text{-Vect}$ and $V := f\bullet$, then $fm \in \mathbb{K}\text{-Vect}(V, V)$ for every $m \in M$, that is, f is a map $M \to \mathbb{K}\text{-Vect}(V, V)$. Moreover, as f is a functor, it is the case that f sends the neutral element of M to 1_V and that $f(mn) = f(m) \circ f(n)$ for every $m, n \in M$. Hence f is indeed a monoid homomorphism $M \to \mathbb{K}\text{-Vect}(V, V)$ i.e. a $\mathbb{K}$-linear representation of M on V. Viceversa, if f is a $\mathbb{K}$-linear representation of M on V, then f is a functor $M \to \mathbb{K}\text{-Vect}$ sending the unique object $\bullet$ of M into the object V of $\mathbb{K}\text{-Vect}$. Indeed $f(m) \in \mathbb{K}\text{-Vect}(V, V)$ for every $m \in M$. Moreover, being a monoid homomorphism, f sends the neutral element of M (i.e. the identity arrow $1_\bullet$) to the one of $\mathbb{K}\text{-Vect}(V, V)$

(i.e. the identity arrow 1_V). Moreover, being a monoid homomorphism, f commutes with the operations of the monoids M and $\mathbb{K}$-$\textsc{Vect}(V, V)$, that is, the relation f converts the composition of the category M into the one of $\mathbb{K}$-$\textsc{Vect}$. Hence f is a functor $M \to \mathbb{K}$-$\textsc{Vect}$. Hence we proved that:

Fact 1.8. *Whenever M is a monoid (or a group), the linear representations of M are precisely the functors $M \to \mathbb{K}$-$\textsc{Vect}$.*

Digression. As Fact 1.8 suggests, we may also consider arrows between two different $\mathbb{K}$-linear representations $f, g\colon M \to \mathbb{K}$-$\textsc{Vect}$ of a given monoid (or group) M. Obviously these should be the natural transfomations $f \to g$. As the unique object of M is $\bullet$, a natural transfomation $f \to g$ is nothing but an arrow $\alpha\colon V := f\bullet \to g\bullet =: W$ of $\mathbb{K}$-$\textsc{Vect}$ (that is, a $\mathbb{K}$-linear map $\alpha\colon V \to W$) such that, for every element $m \in M$, it is the case that $\alpha \circ fm = gm \circ \alpha$. Hence up to now we have given an explicit description of the category $\textsc{Cat}(M, \mathbb{K}\text{-}\textsc{Vect})$, whose objects are the functors $M \to \mathbb{K}$-$\textsc{Vect}$ i.e. the $\mathbb{K}$-linear representations of M and whose arrows are the natural transformations between them.

We may also generalise the notion of representation by considering *representations* of M in other categories. For instance, if we consider the category $\textsc{Set}$ instead of $\mathbb{K}$-$\textsc{Vect}$, we obtain the category $\textsc{Cat}(M, \textsc{Set})$ whose objects are the representations of M in a set i.e. the left M-actions.

Finally we generalise the notion of representation of a monoid (group) to an arbitrary symmetric monoidal category:

Definition 1.9. Let $(\mathcal{C}, \otimes, \eta, \tau)$ be a symmetric monoidal category and let $\mathbb{K}$ be a field. A $\mathbb{K}$-*linear representation of* $(\mathcal{C}, \otimes, \eta, \tau)$ is a symmetric monoidal functor:

$$(\mathcal{C}, \otimes, \eta, \tau) \to (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma).$$

The category whose objects are the $\mathbb{K}$-linear representations of $(\mathcal{C}, \otimes, \eta, \tau)$ and whose arrows are the monoidal natural transformations between them, i.e. the category:

$$\textsc{SymMonCat}((\mathcal{C}, \otimes, \eta, \tau), (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma)),$$

is indicated as $\mathbb{K}$-$\textsc{linRep}(\mathcal{C}, \otimes, \eta, \tau)$ and is called *category of the* $\mathbb{K}$-*linear representation of* $(\mathcal{C}, \otimes, \eta, \tau)$.

2 Cobordism Categories

Whenever X is a smooth compact n-dimensional manifold for some $n \in \mathbb{N} \setminus \{0\}$, we denote its boundary as ∂X with the induced structure that makes it a smooth compact $(n-1)$-dimensional manifold without boundary (see [3] - Section 2.3 or [4]). If X has an orientation, then ∂X denotes its boundary with the induced orientation (see [3] - Section 2.11 or [4]).

Let $n \in \mathbb{N} \setminus \{0\}$. If S and S' are smooth compact $(n-1)$-dimensional manifolds without boundary, a *cobordism* of S and S' is a smooth compact n-dimensional manifold M such that $\partial M = S \sqcup S'$. If S and S' have an orientation, we say that M is an *oriented cobordism* from S to S' if $\partial M = \overline{S} \sqcup S'$, being $\overline{S}$ the manifold S with opposite orientation (we remind that an orientable smooth compact manifold admits precisely two orientations; see [3] - Section 2.11 or [4]). In this case, we also denote $\partial M_A := \overline{S}$ and $\partial M_B := S'$. Observe that, given an oriented compact smooth n-dimensional manifold, there are a priori different notions of ∂M_A and ∂M_B, depending on the choice of S and S' such that $\partial M = \overline{S} \sqcup S'$. In order to avoid any ambiguity, we will only use this notation when the choice of S and S' is clear. Whenever M and M' are oriented cobordisms from S to S', we say that M and M' are *equivalent* if and only if there is an orientation preserving diffeomorphism $\varphi\colon M \to M'$ commuting with the inclusions of $\overline{S}$ and S' into M and M' respectively (we remind that the inclusions $\overline{S} \hookrightarrow \partial M$ $(\partial M')$ and $S' \hookrightarrow \partial M$ $(\partial M')$ are orientation preserving smooth maps), that is, φ is such that the following diagram:

$$\begin{tikzcd}
\partial M_A \arrow[r, hook] \arrow[d, equal] & M \arrow[dd, "\varphi"] & \partial M_B \arrow[l, hook'] \arrow[d, equal] \\
\overline{S} \arrow[d, equal] & & S' \arrow[d, equal] \\
\partial M'_A \arrow[r, hook] & M' & \partial M'_B \arrow[l, hook']
\end{tikzcd}$$

commutes. This relation of *equivalence* between oriented cobordisms with same source and target actually defines an equivalence relation.

A motivated generalisation of the notion of oriented cobordism will simplify some of the following arguments. Let S and S' be oriented smooth compact $(n-1)$-dimensional manifolds without boundary and let M be an oriented smooth compact n-dimensional manifold. Let us assume that there are two orientation preserving diffeomorphisms $\overline{S} \xrightarrow{\cong} \partial M_A$ and $S' \xrightarrow{\cong} \partial M_B$ and morevoer let us assume that $\partial M = \partial M_A \sqcup \partial M_B$. Then we say that the triple $(M, \overline{S} \xrightarrow{\cong} \partial M_A, S' \xrightarrow{\cong} \partial M_B)$ is a *generalised oriented cobordism* from S to S'. Clearly any oriented cobordism from S to S' with the two identities $\overline{S} = \partial M_A$ and $S' = \partial M_B$ is also a generalised oriented cobordism from S to S'. Viceversa, one can prove (see [7]) the following:

Fact 2.1. *Whenever $(M, \overline{S} \xrightarrow{\cong} \partial M_A, S' \xrightarrow{\cong} \partial M_B)$ is a generalised oriented cobordism from S to S', then there are an oriented cobordism M' from S to S' and an orientation preserving* *diffeomorphism* $\varphi\colon M \to M'$ *such that the following diagram:*

$$\begin{tikzcd} \partial M_A \arrow[r, hook] & M \arrow[dd, "\varphi"] & \partial M_B \arrow[l, hook'] \\ \overline{S} \arrow[u, equal] \arrow[d, "\cong"] & & S' \arrow[u, equal] \arrow[d, "\cong"] \\ \partial M'_A \arrow[r, hook] & M' & \partial M'_B \arrow[l, hook'] \end{tikzcd}$$

commutes.

As we are going to see, this fact motivates the generalisation.

Whenever $(M, \overline{S} \xrightarrow{\cong} \partial M_A, S' \xrightarrow{\cong} \partial M_B)$ and $(M', \overline{S} \xrightarrow{\cong} \partial M'_A, S' \xrightarrow{\cong} \partial M'_B)$ are generalised oriented cobordisms form S to S', we say that M and M' are *equivalent* if and only if there is an orientation preserving diffeomorphism $\varphi\colon M \to M'$ such that the following diagram:

$$\begin{tikzcd} \partial M_A \arrow[r, hook] & M \arrow[dd, "\varphi"] & \partial M_B \arrow[l, hook'] \\ \overline{S} \arrow[u, "\cong"] \arrow[d, "\cong"] & & S' \arrow[u, "\cong"] \arrow[d, "\cong"] \\ \partial M'_A \arrow[r, hook] & M' & \partial M'_B \arrow[l, hook'] \end{tikzcd}$$

commutes. Again, this relation of *equivalence* between generalised oriented cobordisms with same source and target defines an equivalence relation. Moreover this equivalence relation contains (i.e. extends) the previous one, that is, coincides with the previous one once restricted to the class of the oriented cobordisms. The Fact 2.1 says that every generalised oriented diffeomorphism is equivalent to an oriented cobordism. Hence it justifies the generalisation of the notions of *oriented cobordisms* and *equivalence* between them. We are ready to present the definition of the so-called *cobordism categories*, that will constitute a fundamental instance of the notion of monoidal category.

Let $n \in \mathbb{N} \setminus \{0\}$. Let $n\textsc{Cob}$ be the category:

1. whose objects are the oriented smooth compact $(n-1)$-dimensional manifolds without boundary;

2. (given two objects S and S') whose morphisms $S \to S'$ are the equivalence classes of generalised oriented cobordisms from S to S' (according to the previously defined notion of equivalence between generalised oriented cobordisms with same source and target);

3. (given two morphisms $[M]\colon S \to S'$ and $[M']\colon S' \to S''$) whose notion of composition $[M'] \circ [M]\colon S \to S''$ is given by the class of cobordisms $[M'M]$, whose representative $M'M$ is the generalised oriented cobordism from S to S'' given by the gluing of M and M' through the orientation preserving diffeomorphism:

$$(S' \xrightarrow{\cong} \overline{\partial M'_A}) \circ (S' \xrightarrow{\cong} \partial M_B)^{-1}\colon \partial M_B \xrightarrow{\cong} \overline{\partial M'_A};$$

4. (given an object S) whose notion of identity morphism is given by the class $1_S \colon S \to S$ represented by the cylinder $S \times [0,1]$ with one of the two orientations (observe that there is an orientation preserving diffeomorphism $S \times [0,1] \to \overline{S \times [0,1]}$ commuting with the inclusions).

One can verify that these data actually define a category. The categories $n\mathrm{Cob}$, for $n \in \mathbb{N} \setminus \{0\}$, are called *cobordism categories*.

Remark 2.2. Let $n \in \mathbb{N} \setminus \{0\}$. Then the functor $\sqcup \colon n\mathrm{Cob} \times n\mathrm{Cob} \to n\mathrm{Cob}$ sending a morphism $([M],[M']) \colon (S,S') \to (T,T')$ of $n\mathrm{Cob} \times n\mathrm{Cob}$ to the morphism $[M \sqcup M'] \colon S \sqcup S' \to T \sqcup T'$ and the functor $\mathtt{1} \to n\mathrm{Cob}$ such that $* \mapsto \emptyset$ (observe that the empty set is an oriented smooth compact $(n-1)$-dimensional manifold without boundary) constitute a monoidal structure over $n\mathrm{Cob}$.

Observe that, in general, given two objects S, S' of $n\mathrm{Cob}$, it is not the case that $S \sqcup S' = S' \sqcup S$. Indeed, this happens if and only if $S = S'$. That happens because, given two sets A, B, then $A \sqcup B := \{(x,1) : x \in A\} \cup \{(x,2) : x \in B\}$. Anyway, there is a bijection $\tau_{A,B} \colon A \sqcup B \xrightarrow{\cong} B \sqcup A$ such that $\tau_{A,B}(a,1) = (a,2)$ and $\tau_{A,B}(b,2) = (b,1)$ for every $a \in A$ and every $b \in B$. Clearly $\tau_{S,S'}$ is also an orientation preserving isomorphism.

Basic facts about the cobordism categories

Let us present some basic properties of $n\mathrm{Cob}$.

Remark 2.3. As any representative (that is, a generalised oriented cobordism) of a morphism in a cobordism category is equivalent to an oriented cobordism, it is the case that such a morphism is also represented by an oriented cobordism. Therefore, whenever we consider such a representative, without loss of generality we may assume it to be an oriented cobordism (that is, the two order preserving diffeomorphisms constituting it are identities). This is for instance what we do in the proof of Proposition 2.5.

Remark 2.4. Let $n \in \mathbb{N}$. Observe that, since any object of $n\mathrm{Cob}$ is assumed to be a compact manifold, then its connected components need to be finitely many. The same holds for any representative of a given morphism in $n\mathrm{Cob}$, as it is also required to be compact.

Proposition 2.5. *Let S and T be objects of $1\mathrm{Cob}$. Let $s_+ \in \mathbb{N}$ be the number of the positively oriented connected components (i.e. points) of S, let $s_- \in \mathbb{N}$ be the number of the negatively oriented connected components (i.e. points) of S, let $t_+ \in \mathbb{N}$ be the number of the positively oriented connected components (i.e. points) of T and let $t_- \in \mathbb{N}$ be the number of the negatively oriented connected components (i.e. points) of T. Then there is an arrow $S \to T$ of $1\mathrm{Cob}$ if and only if $s_+ + t_- = s_- + t_+$.*

Proof. Let us assume that there is an arrow $[M] \colon S \to T$. Then M is a (finite) disjoint union of oriented circumferences and oriented segments. Let A be the set whose elements are the oriented segments being connected components of M. Obviously ∂M is also the boundary of the disjoint union of the elements of A, as a circumference has empty boundary. Hence, let us consider the following partition of A: let A_1 be the set whose elements are the elements of A whose beginning point is a positively oriented point of S and whose ending point is a positively oriented point of T; let A_2 be the set whose elements are the elements of A whose beginning point is a negatively oriented point of T and whose ending point is a negatively oriented point of S; let A_3 be the set whose elements are the elements of A whose beginning point is a positively oriented point of S and whose ending point is a negatively oriented point of S; let A_4 be the set whose elements are the elements of A whose beginning point is a negatively oriented point of T and whose ending point is a positively oriented point of T.z Hence $s_+ = |A_1| + |A_3|$, $s_- = |A_2| + |A_3|$, $t_+ = |A_1| + |A_4|$ and $t_- = |A_2| + |A_4|$ and therefore $s_+ + t_- = |A| = s_- + t_+$.

Viceversa, let us assume that $s_+ + t_- = s_- + t_+$. Without loss of generality, let us assume that $x := s_+ - t_+ = s_- - t_- \geq 0$. Let us consider an arbitrary partition of the positively oriented points of S into a set S_1^+ of cardinality $s_+ - x = t_+$ and a set S_2^+ of cardinality x. Ananogously, let us consider a partition of the negatively oriented points of S into a set S_1^- of cardinality $s_- - x = t_-$ and a set S_2^- of cardinality x. Finally, let T^+ be the set of the positively oriented points of T and let T^- be the set of its negatively oriented points. Let us consider a bijection $f\colon S_1^+ \to T^+$, a bijection $g\colon S_1^- \to T^-$ and a bijection $h\colon S_2^+ \to S_2^-$. For every $(a, b) \in f \cup g$ (if this notation is not immediately clear, see Appendix **4.**) let us consider an oriented segment whose boundary is $\overline{a} \sqcup b$ and for every $(a, b) \in h$ let us consider an oriented segment whose boundary is $\overline{a} \sqcup \overline{b}$. The disjoint union of these segments is therefore an oriented smooth compact 1-dimensional manifold whose boundary is $\overline{S} \sqcup T$. Q.E.D.

Observe that Proposition 2.5 is enough to give an explicit description (a classification) of the unidimensional TQFTs w.r.t. a given field (see Definition 2.11).

Proposition 2.6. *For every choice of objects S and T of* 2Cob *there is an arrow $S \to T$. In particular, the category* 2Cob *is connected.*

Proof. As S and T are oriented smooth compact 1-dimensional manifolds without boundary, they are finite disjoint unions of oriented circumferences. Let us consider, for every connected component of S, an oriented semisphere whose boundary is that connected component with the opposite orientation. Moreover, let us consider, for every connected component of T, an oriented semisphere whose boundary is that connected component. The disjoint union of these semispheres is therefore an oriented smooth compact 2-dimensional manifold whose boundary is $\overline{S} \sqcup T$. Q.E.D.

Remark 2.7. Let S be an object of nCob, for some $n \in \mathbb{N}$. As the cylinder $\mathfrak{C}S := S \times [0, 1]$ with a given orientation is a representative of 1_S and in particular its boundary is $\partial\mathfrak{C}S = \overline{S} \sqcup S$, actually it is a representative of different morphisms of nCob, depending on how we interpret $\partial\mathfrak{C}S_A$ and $\partial\mathfrak{C}S_B$. Considering $\partial\mathfrak{C}S_A := \overline{S}$ and $\partial\mathfrak{C}S_B := S$, we can as usual regard $\mathfrak{C}S$ as a representative of the morphism $1_S\colon S \to S$. However if we consider $\partial\mathfrak{C}S_A := \overline{S} \sqcup S$ and $\partial\mathfrak{C}S_B := \emptyset$, we see that $\mathfrak{C}S$ is also a representative of a morphism $S \sqcup \overline{S} \to \emptyset$ that we indicate as lU. Moreover, if $\partial\mathfrak{C}S_A := \emptyset$ and $\partial\mathfrak{C}S_B := \overline{S} \sqcup S$, then we get a representative of a morphism $rU\colon \emptyset \to \overline{S} \sqcup S$.

Reminding that $\sqcup$ is a functor, we get a morphism $1_S \sqcup rU\colon S \to S \sqcup \overline{S} \sqcup S$ and a morphism $lU \sqcup 1_S\colon S \sqcup \overline{S} \sqcup S \to S$. Hence, a morphism $(lU \sqcup 1_S) \circ (1_S \sqcup rU)\colon S \to S$. By the notion of composition of nCob, a representative of $(lU \sqcup 1_S) \circ (1_S \sqcup rU)$ is $S \times ([0, 1] \sqcup [0', 1'] \sqcup [0'', 1''] \sqcup [0''', 1'''])/_\sim$ where the equivalence relation $\sim$ is the smallest one such that $1 \sim 0'$, $1' \sim 0''$ and $1'' \sim 0'''$. Hence $S \times ([0, 1] \sqcup [0', 1'] \sqcup [0'', 1''] \sqcup [0''', 1'''])/_\sim \cong S \times [0, 1]$ and therefore $(lU \sqcup 1_S) \circ (1_S \sqcup rU) = 1_S$.

Proposition 2.8. *Let $n \in \mathbb{N}$ and let $[M]\colon S \to S'$ be an isomorphism of* nCob. *Let us assume that M is connected. Then S and S' are connected as well.*

Proof. Let N be such that $[N] = [M]^{-1}\colon S' \to S$. Then $[NM] = 1_S$. Then there is a diffeomorphism $NM \cong S \times [0,1]$. As for every $p \in S \times [0,1]$ there is $p' \in \partial(S \times [0,1])_A = S$ such that p and p' are in the same connected component of $S \times [0,1]$, this property must also hold for the manifold NM. Hence, for every $p \in NM$ there is $p' \in \partial(NM)_A = \partial M_A = S$ such that p and p' are in the same connected component of NM. Being M connected, $\partial M_A = S$ is contained in the unique connected component of M, hence all points of NM need to be in the same connected component of NM, that is, NM is connected. Then $S \times [0,1]$ is connected and therefore S needs to be connected as well: if by contradiction $S = S_1 \sqcup S_2$, being S_1 and S_2 closed and nonempty, then $S = (S_1 \times [0,1]) \sqcup (S_2 \times [0,1])$ and $S_1 \times [0,1]$ and $S_2 \times [0,1]$ are closed and nonempty. The same argument proves that S' is also connected. Q.E.D.

Remark 2.9. Let us consider an order preserving diffeomorphism $\varphi\colon S \to S'$ between oriented compact smooth $(n-1)$-dimensional manifolds without boundary. Let us consider the manifold $S' \times [0,1]$ with an orientation and let us consider $\partial(S' \times [0,1])_A := \overline{S'}$ and $\partial(S' \times [0,1])_B := S'$, so that $S' \times [0,1]$ is a representative of $1_{S'}$. As there is an orientation preserving diffeomorphism $\overline{\varphi}\colon \overline{S} \to \overline{S'}$, the triple:

$$(S' \times [0,1], \overline{\varphi}\colon \overline{S} \to \overline{S'} = \partial(S' \times [0,1])_A, S' = S' = \partial(S' \times [0,1])_B)$$

is a representative of an arrow $[M^\varphi]\colon S \to S'$ of nCob. In other words (use Fact 2.1), there are an oriented compact smooth n-dimensional manifold M^φ and an orientation preserving diffeomorphism $f\colon M^\varphi \to S' \times [0,1]$ such that $\partial M^\varphi_A = \overline{S}$, $\partial M^\varphi_B = S'$, $(M^\varphi \xrightarrow{f} S' \times [0,1]) \circ (\overline{S} \hookrightarrow M^\varphi) = (\overline{S}' \hookrightarrow S' \times [0,1]) \circ (\overline{S} \xrightarrow{\overline{\varphi}} \overline{S'})$ and $(M^\varphi \xrightarrow{f} S' \times [0,1]) \circ (S' \hookrightarrow M^\varphi) = (S' \hookrightarrow S' \times [0,1])$. Observe that $[M^\varphi]$ is also represented by the triple:

$$(S \times [0,1], \overline{S} = \overline{S} = \partial(S \times [0,1])_A, \varphi^{-1}\colon S' \to S = \partial(S \times [0,1])_B).$$

One can verify that this construction defines a functor $[M^-]$ from the groupoid of the oriented compact smooth $(n-1)$-dimensional manifolds without boundary and the orientation preserving diffeomorphisms to nCob, that is, $[M^{\psi\circ\varphi}] = [M^\psi] \circ [M^\varphi]$, for any composable orientation preserving diffeomorphisms φ and ψ, and $[M^-]$ sends identity maps of given oriented compact smooth $(n-1)$-dimensional manifolds without boundary to their identity morphisms in nCob. Hence in particular it is the case that M^φ is an isomorphism of nCob, whose inverse is $M^{\varphi^{-1}}$.

Moreover, given two orientation preserving diffeomorphisms $\varphi, \psi\colon S \to S'$, it is the case that $[M^\varphi] = [M^\psi]$ if and only if φ and ψ are smoothly homotopic, that is, there is a smooth map $h\colon S \times [0,1] \to S'$ such that $h(-,0) = \varphi$ and $h(-,1) = \psi$. Indeed, if such a homotopy h exists, then the following diagram:

$$\begin{tikzcd}
\overline{S} \arrow[r, hook] \arrow[d, equal] & S \times [0,1] \arrow[dd, "{\langle h,\ \mathrm{id}_{[0,1]} \circ \pi_{[0,1]} \rangle}"] & S \arrow[l, hook'] \\
\overline{S} \arrow[d, "\overline{\psi}"] & & S' \arrow[u, "\varphi^{-1}"'] \arrow[d, equal] \\
\overline{S'} \arrow[r, hook] & S' \times [0,1] & S' \arrow[l, hook']
\end{tikzcd}$$

commutes. Viceversa, if such a diagram:

$$\begin{tikzcd}
\overline{S} \arrow[r, hook] \arrow[d, equal] & S \times [0,1] \arrow[dd, "\Phi"] & S \arrow[l, hook'] \\
\overline{S} \arrow[d, "\overline{\psi}"] & & S' \arrow[u, "\varphi^{-1}"'] \arrow[d, equal] \\
\overline{S'} \arrow[r, hook] & S' \times [0,1] & S' \arrow[l, hook']
\end{tikzcd}$$

commutes for some orientation preserving diffeomorphism Φ, then $h := \pi_{S'} \circ \Phi$ is a smooth homotopy $\psi \simeq \varphi$.

Therefore $[M^{-}]$ becomes a functor from the quotient groupoid whose objects are the ones of $n\text{Cob}$ and whose arrows are the equivalence classes of smoothly homotopic diffeomorphisms between them to $n\text{Cob}$.

Definition of TQFT

Remark 2.10 (*Symmetric monoidal structure over* $n\text{Cob}$)**.** Let $n \in \mathbb{N} \setminus \{0\}$ and let us consider the monoidal category $(n\text{Cob}, \sqcup, \emptyset)$ of Remark 2.2. For every object (S, T) of $n\text{Cob} \times n\text{Cob}$, we define $\tau_{(S,T)}$ to be the isomorphism $S \sqcup T \to T \sqcup S$ of $n\text{Cob}$ induced by the orientation preserving diffeomorphism $\tau_{S,T} \colon S \sqcup T \to T \sqcup S$ of Remark 2.2 through the procedure inlustrated in Remark 2.9. In other words, it is the case that $\tau_{(S,T)} := [M^{\tau_{S,T}}]$. Actually the isomorphism $\tau_{(S,T)}$ is natural in (S,T), that is, for every morphism $(f, g) \colon (S, T) \to (S', T')$ of $n\text{Cob} \times n\text{Cob}$, the following square:

$$\begin{tikzcd}
S \sqcup T \arrow[r, "\tau_{(S,T)}"] \arrow[d, "f \sqcup g"'] & T \sqcup S \arrow[d, "g \sqcup f"] \\
S' \sqcup T' \arrow[r, "\tau_{(S',T')}"] & T' \sqcup S'
\end{tikzcd}$$

commutes. In other words, τ is a natural isomorphism $\sqcup \to \sqcup \circ (\pi_2 \times \pi_1)$. Hence $(n\text{Cob}, \sqcup, \emptyset, \tau)$ is a symmetric monoidal category.

We are ready to give the following:

Definition 2.11 (*TQFT w.r.t.* $\mathbb{K}$)**.** Let $n \in \mathbb{N} \setminus \{0\}$ and let $\mathbb{K}$ be a field. Then an *n-dimensional topological quantum field theory (TQFT) w.r.t.* $\mathbb{K}$ is a symmetric monoidal functor $(n\text{Cob}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-Vect}, \otimes, \mathbb{K}, \sigma)$, that is, a $\mathbb{K}$-linear representation of the symmetric monoidal category $n\text{Cob}$.

We denote as $n\text{TQft}(\mathbb{K})$ the category whose objects are the n-dimensional TQFTs w.r.t. $\mathbb{K}$ and whose arrows are the monoidal natural transformations between them. In other words:

$$\begin{aligned}
n\text{TQft}(\mathbb{K}) &= \mathbb{K}\text{-linRep}(n\text{Cob}, \sqcup, \emptyset, \tau) \\
&= \text{SymMonCat}((n\text{Cob}, \sqcup, \emptyset, \tau), (\mathbb{K}\text{-Vect}, \otimes, \mathbb{K}, \sigma))
\end{aligned}$$

(see Definition 1.9).

Proposition 2.12. *Let $n \in \mathbb{N} \setminus \{0\}$ and let $\mathbb{K}$ be a field. Let F be an n-dimensional TQFT w.r.t. $\mathbb{K}$. Then for every object S of $n\text{Cob}$ it is the case that FS is a finite dimensional $\mathbb{K}$-linear space.*

Proof. Let S be an object of nCob and let us consider the corresponding arrows $lU\colon S\sqcup\overline{S}\to\emptyset$ and $rU\colon\emptyset\to\overline{S}\sqcup S$ of Remark 2.7. We know that $(lU\sqcup 1_S)\circ(1_S\sqcup rU)=1_S$. Being F a symmetric monoidal functor $(n\textsc{Cob},\sqcup,\emptyset,\tau)\to(\mathbb{K}\text{-}\textsc{Vect},\otimes,\mathbb{K},\sigma)$, it preserves the compositions and converts the applications of $\sqcup$ into applications of $\otimes$. Hence, if $f:=F(lU)\colon FS\otimes F\overline{S}\to F\emptyset=\mathbb{K}$ and $g:=F(rU)\colon\mathbb{K}=F\emptyset\to F\overline{S}\otimes FS$, it is the case that:

$$(FS\xrightarrow{1_{FS}}FS)=(FS=FS\otimes\mathbb{K}\xrightarrow{1_{FS}\otimes g}FS\otimes F\overline{S}\otimes FS\xrightarrow{f\otimes 1_{FS}}\mathbb{K}\otimes FS=FS),$$

that in particular implies that f and g are not null maps (otherwise 1_{FS} would be the null map $FS\to FS$). Therefore there are $k\in\mathbb{N}\setminus\{0\}$ and $x_i\in F\overline{S}\setminus\{0\}$ and $y_i\in FS\setminus\{0\}$ for $i\in\{1,...,k\}$ such that $g(1)=\sum_{i=1}^{k}(x_i\otimes y_i)$, being 1 the neutral element of $\mathbb{K}$ w.r.t. its multiplication. Now, let $y\in FS$. Then:

$$\begin{aligned}
y &= 1_{FS}(y)=(f\otimes 1_{FS})\circ(1_{FS}\otimes g)(y)\\
[FS=FS\otimes\mathbb{K}] &= (f\otimes 1_{FS})\circ(1_{FS}\otimes g)(y\otimes 1)\\
&= (f\otimes 1_{FS})(y\otimes g(1))=(f\otimes 1_{FS})(y\otimes\sum_{i=1}^{k}(x_i\otimes y_i))\\
&= (f\otimes 1_{FS})(\sum_{i=1}^{k}(y\otimes x_i\otimes y_i))=\sum_{i=1}^{k}(f\otimes 1_{FS})(y\otimes x_i\otimes y_i)\\
&= \sum_{i=1}^{k}(f(y\otimes x_i)\otimes y_i)=\sum_{i=1}^{k}f(y\otimes x_i)(1\otimes y_i)\\
[\mathbb{K}\otimes FS=FS] &= \sum_{i=1}^{k}f(y\otimes x_i)y_i
\end{aligned}$$

and being $y\in FS$ arbitrary, it is the case that $\{y_i\}_{i=1}^{k}$ is a generating set of FS. Hence FS is finite dimensional. Q.E.D.

As expressed in the Introduction, our aim is to describe the 2-dimensional TQFTs w.r.t. a given field $\mathbb{K}$. We will also give an explicit description of $2\textsc{TQft}(\mathbb{K})$. Firstly, let us briefly explain why a topological quantum field theory should be defined in such a way (according to [6, 7, 9] - see [5] for further details). In our formulation if we assume the spacetime to be n-dimensional, then the objects of nCob represent the *space* and the arrows between them the *space-time*. An n-dimensional TQFT w.r.t. $\mathbb{K}$ is just a way to assign to any system, i.e. to any object of nCob, a space of admissible states for the given system and to any space-time between two systems, i.e. to any arrow of nCob, an operator representing the time evolution between the spaces of the admissible states.

The fact that a TQFT is a functor (i.e. preserves compositions and identities) means two things: the passage of time represented by a given class of cobordisms followed by the passage of time represented by another class of cobordisms has, on the time evolution between the spaces of the admissible states, the same consequences that the passage of time represented by the composition of the two given classes of cobordisms has; the null passage of time represented by an identity cobordism class (i.e. an identity arrow of nCob) corresponds to a null time evolution of the space of the admissible states.

Finally the fact that a TQFT is monoidal turns into a fundamental principle of Quantum Mechanics expressing that the space of the admissible states of a system made of two independent systems (i.e. the disjoint union of two objects of nCob) is the tensor product of the corresponding spaces of the admissible states of the two independent systems that we started from.

Before concluding this section, let us prove one result more:

Proposition 2.13. *Let $[T]\colon X \to Y$ be an isomorphism of nCob for some $n \in \mathbb{N} \setminus \{0\}$and let us assume that:*

$$([T]\colon X \to Y) = ([T']\colon X' \to Y') \sqcup ([T'']\colon X'' \to Y'')$$

for some morphisms $[T']$ and $[T'']$ of nCob. Then $[T']$ and $[T'']$ are isomorphisms as well. In other words, the functor $\sqcup\colon n\mathrm{Cob} \times n\mathrm{Cob} \to n\mathrm{Cob}$ reflects the isomorphisms.

Proof. Let $[S]\colon Y \to X$ be such that $[ST] = 1_X$ and $[TS] = 1_Y$. Without loss of generality let us assume that $ST = X \times [0,1] = (X' \times [0,1]) \sqcup (X'' \times [0,1])$ and $TS = Y \times [0,1] = (Y' \times [0,1]) \sqcup (Y'' \times [0,1])$. As $T' \sqcup T'' = T \subseteq ST = (X' \times [0,1]) \sqcup (X'' \times [0,1])$ and $\overline{\partial T'_A} = X'$ and $\overline{\partial T''_A} = X''$, it is the case that $T' \subseteq X' \times [0,1]$ and $T'' \subseteq X'' \times [0,1]$. Analogously, as $T' \sqcup T'' = T \subseteq TS = (Y' \times [0,1]) \sqcup (Y'' \times [0,1])$ and $\partial T'_B = Y'$ and $\partial T''_B = Y''$ it is the case that $T' \subseteq Y' \times [0,1]$ and $T'' \subseteq Y'' \times [0,1]$.

If $X' = \emptyset$ then $T' \subseteq X' \times [0,1] = \emptyset$, hence $T' = \emptyset$ and $[T']$ is an isomorphism. Then $[T''] = [T]$ is an isomorphism as well. The same holds if $Y' = \emptyset$. Analogously, if $X'' = \emptyset$ of $Y'' = \emptyset$ it is the case that $T'' = \emptyset$, hence $[T'']$ is an isomorphism and $[T'] = [T]$ is an isomorphism as well.

Therefore we can assume $X', Y', X'', Y'' \neq \emptyset$, otherwise we are done. Then T' and T'' need to be nonempty, otherwise $X', Y' = \emptyset$ (if $T' = \emptyset$) and $X'', Y'' = \emptyset$ (if $T'' = \emptyset$).

As $Y' \neq \emptyset$ and $Y'' \neq \emptyset$ and as $\overline{\partial S_A} = \overline{\partial (TS)_A} = \overline{\partial (Y \times [0,1])_A} = Y = Y' \sqcup Y''$, it is the case that $S' := S \cap (Y' \times [0,1]) \neq \emptyset$ and $S'' := S \cap (Y'' \times [0,1]) \neq \emptyset$. As $S \subseteq Y \times [0,1]$, it is the case that $S = S' \sqcup S''$. Hence $Y' \times [0,1] \sqcup Y'' \times [0,1] = TS = T'S' \sqcup T''S''$ and, since $\partial T'_B = Y'$ and $\partial T''_B = Y''$, it must be the case that $T'S' = Y' \times [0,1]$ and $T''S'' = Y'' \times [0,1]$. In particular $\overline{\partial S'_A} = \overline{\partial (T'S')_A} = Y'$ and $\overline{\partial S''_A} = \overline{\partial (T''S'')_A} = Y''$.

Analogously, as $X' \neq \emptyset$ and $X'' \neq \emptyset$ and as $\partial S_B = \partial (ST)_B = \partial (X \times [0,1])_A = X = X' \sqcup X''$, it is the case that $\Sigma' := S \cap (X' \times [0,1]) \neq \emptyset$ and $\Sigma'' := S \cap (X'' \times [0,1]) \neq \emptyset$. As $S \subseteq X \times [0,1]$, it is the case that $S = \Sigma' \sqcup \Sigma''$. By contradiction, if $S' \subseteq \Sigma''$, that is, $S' = \Sigma''$ then $S \cap (Y' \times [0,1]) = S \cap (X'' \times [0,1])$. Hence $X' \times [0,1] \sqcup X'' \times [0,1] = ST = \Sigma' T' \sqcup \Sigma'' T''$ and, since $\overline{\partial T'_A} = X'$ and $\overline{\partial T''_A} = X''$, it must be the case that $\Sigma' T' = X' \times [0,1]$ and $\Sigma'' T'' = X'' \times [0,1]$. In particular $\partial \Sigma'_B = \partial (\Sigma' T')_B = Y'$ and $\partial \Sigma''_B = \partial (\Sigma'' T'')_B = Y''$.

Either $\Sigma' = S'$ or $\Sigma' = S''$. If by contradiction the latter holds, then $\partial S''_B = \partial \Sigma'_B = Y'$ and the equality $T''S'' = Y'' \times [0,1]$ does not make sense, as the first member does not exist. Then $\Sigma' = S'$ and $\Sigma'' = S''$. This concludes that $[S']$ is the inverse of $[T']$ and $[S'']$ is the inverse of $[T'']$. Q.E.D.

3 An explicit description of 2Cob...

...in terms of (a class of) generators and (a class of) relations.

In order to give an explicit description of the 2-dimensional TQFTs, at first we need an explicit description of their domain category 2Cob. A useful and exhaustive description of a given category consists of a so-called *presentation* of it.

Let $\mathcal{C}$ be a category. Let S be a class of arrows of $\mathcal{C}$ such that every arrow of $\mathcal{C}$ is a (finite) composition of arrows of S. Then we say that S is a *generating class* for $\mathcal{C}$. Obviously such a class S does not need to have the property that, whenever an arrow of $\mathcal{C}$ is a composition of arrows of S, then this composition is unique. Whenever an arrow f of $\mathcal{C}$ is such that $f = h_1 \circ ... \circ h_n$ and $f = k_1 \circ ... \circ k_m$ for some $n, m \in \mathbb{N}$ and some arrows h_i, k_i of S, then the equality $h_1 \circ ... \circ h_n = k_1 \circ ... \circ k_m$ is a *relation for* S. Let us assume that R is a class of relations for S such that any given relation for S can be deduced by the elements of R. Then we say that R is *complete*. Finally, a *presentation* of $\mathcal{C}$ is a couple (S, R), where S is a generating class for $\mathcal{C}$ and R is a complete class of relations for S.

Hence the aim of this section is to find an opportune presentation of 2Cob. Actually, as TQFT's are functors *from* 2Cob and thus preserve isomorphisms, it is enough to find a presentation of a *skeleton* (see Appendix **3.**) of 2Cob, i.e. a full subcategory $\mathcal{S}$ of 2Cob such that any two isomorphic objects of $\mathcal{S}$ are equal and such that every object of 2Cob is isomorphic to an object of $\mathcal{S}$ (to say the latter property corresponds to saying that the fully faithful embedding $\mathcal{S} \hookrightarrow$ 2Cob is essentially surjective, i.e. an equivalence of categories).

We can also improve the notion of *presentation* for monoidal categories. Let $(\mathcal{C}, \otimes, \eta)$ be a monoidal category and let S be a class of arrows of $\mathcal{C}$ such that every arrow of $\mathcal{C}$ is a (finite) iteration of compositions and $\otimes$-paralleling of arrows of S. Then we say that S is a *generating class* for $(\mathcal{C}, \otimes, \eta)$. Whenever for an arrow f of $\mathcal{C}$ there are two ways of writing f as iteration of compositions and $\otimes$-paralleling of arrows of S, the equality of these two iterations is a *relation for* S. Let us assume that R is a class of relations for S such that any given relation for S can be deduced by the elements of R. Then we say that R is *complete*. Finally:

Definition 3.1. A *(monoidal) presentation* of $(\mathcal{C}, \otimes, \eta)$ is a couple (S, R), where S is a generating class for $(\mathcal{C}, \otimes, \eta)$ and R is a complete class of relations for S.

This section is devoted to the exhibition of a presentation of a skeleton of the symmetric monoidal category $(n\mathrm{Cob}, \sqcup, \emptyset, \tau)$. At first we will need the following:

Theorem 3.2. *Let S and T be objects of* 2Cob*. Then S and T are isomorphic (in* 2Cob*) if and only if they have the same number of connected components.*

whose proof is a consequence of the following Lemma 3.3. Indeed if S and T are isomorphic then, according to Lemma 3.3, they are orientation preservingly diffeomorphic and in particular they have the same number of connected components. Viceversa, suppose that S and T have the same number of connected components and consider an arbitrary bijection f from the set of the connected components of S to the set of the connected components of T. For every $(a, b) \in f$ (if this notation is not immediately clear, see Appendix **4.**) consider an orientation preserving diffeomorphism $g_{(a,b)}: a \to b$: it exists, as a and b are oriented circumferences. Then $\bigsqcup_{(a,b)\in f} g_{(a,b)}$ is an orientation preserving diffeomorphism $S \to T$. Hence S and T are isomorphic by Lemma 3.3. We only need to prove the quoted:

Lemma 3.3. *Two objects S and T of* 2Cob *are isomorphic (in* 2Cob*) if and only if there is an orientation preserving diffeomorphism $S \to T$.*

Proof.

If. This is the content of Remark 2.9.

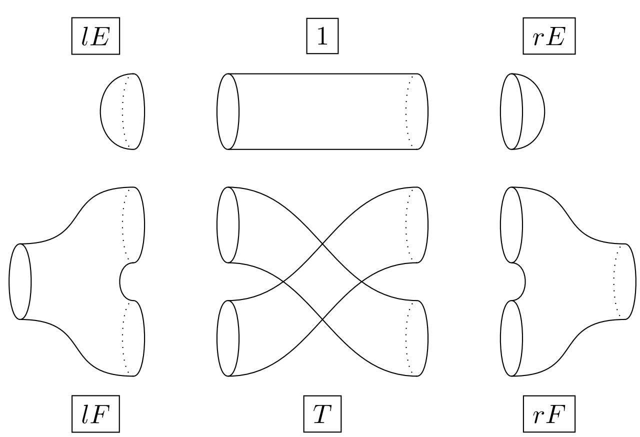
Only if. Let $[M]\colon S \to T$ be an isomorphism of 2Cob. Again, as any two oriented circumferences are orientation preservingly diffeomorphism and S and T are disjoint unions of oriented circumferences, we only need to show that S and T have the same number of connected components. We prove that for every $n \in \mathbb{N}$ it is the case that: $(\#(n))$ if M has n connected components, then S and T have the same number of connected components. If $n = 0$ this is clear and if $n = 1$ this holds because of Proposition 2.8. Let us assume that $n \geq 2$ and that the property $\#(m)$ holds for every $m \leq n-1$. Then, if M has n connected components, it is the case that $M = M' \sqcup M''$ for some manifolds M' and M'' (of the same dimension of M) whose numbers of connected components are n' and n'' respectively and $n', n'' \leq n-1$. Then:

$$([M]\colon S \to T) = ([M']\colon \overline{\partial M'_A} \to \partial M'_B) \sqcup ([M'']\colon \overline{\partial M''_A} \to \partial M''_B)$$

where $\overline{\partial M'_A} \sqcup \overline{\partial M''_A} = S$ and $\partial M'_B \sqcup \partial M''_B = T$. By Proposition 2.13 it is the case that $[M']$ and $[M'']$ are isomorphisms. As $n', n'' \leq n-1$, by inductive hypothesis $\overline{\partial M'_A}$ and $\partial M'_B$ have the same number of connected components and $\overline{\partial M''_A}$ and $\partial M''_B$ have the same number of connected components. Hence S and T have the same number of connected components as well. Hence the property $\#(n)$ holds for every $n \in \mathbb{N}$ and we are done. Q.E.D.

Let $\mathbf{1}$ be a connected object of 2Cob, that is, a circumference with a given orientation. For every $n \geq 2$, let $\boldsymbol{n}$ be the disjoint union of n copies of $\mathbf{1}$. Sometimes we will also denote $\mathbf{0} := \emptyset$. Let $\mathcal{S}$ be the full subcategory of 2Cob whose object class is $\{\emptyset, \mathbf{1}, \mathbf{2}, ...\}$. By Theorem 3.2, it is the case that $\mathcal{S}$ is a skeleton of 2Cob. Observe that $\mathcal{S}$ is closed under the application of the functors $\sqcup$ and $\emptyset$. Hence $(\mathcal{S}, \sqcup, \emptyset, \tau)$ is again a symmetric monoidal category and we can consider monoidal presentations of it.

Theorem 3.4. *Let S be the class whose elements are the arrows represented by the following oriented cobordisms:*



between all the possibile objects of S. In detail: lE (respectively rE) is a representative of the unique arrow $\emptyset \to \mathbf{1}$ (respectively $\mathbf{1} \to \emptyset$) whose representatives are connected; 1 is a *representative of the identity* $\mathbf{1} \to \mathbf{1}$*; lF (respectively rF) is a representative of the unique arrow* $\mathbf{1} \to \mathbf{2}$ *(respectively* $\mathbf{2} \to \mathbf{1}$*) whose representatives are connected; T is a representative of the arrow* $\tau_{(\mathbf{1},\mathbf{1})} \colon \mathbf{2} = \mathbf{1} \sqcup \mathbf{1} \to \mathbf{1} \sqcup \mathbf{1} = \mathbf{2}$ *defined in Remark 2.10 (hence their orientation is determined, as* $\mathbf{1}$ *has a given orientation).*

Then S is a generating class for $(\mathcal{S}, \sqcup, \emptyset, \tau)$*.*

In order to prove Theorem 3.4, we need to show that we are able to build every arrow of $\mathcal{S}$, via (finite) composition and (finite) $\sqcup$-paralleling of the arrows $[lE]$, $[rE]$, $[lF]$, $[rF]$, $[1]$ and $[T]$ of $\mathcal{S}$. By definition of 2Cob, two (generalised) oriented cobordisms with same source and target represent the same arrow if and only if they are diffeomorphic through an orientation preserving diffeomorphism commuting with the inclusions of the source and the target into the given cobordisms. Hence, in order to conclude, for every arrow $[M]$ of $\mathcal{S}$, we need to build, using lE, rE, lF, rF, 1 and T, an oriented cobordism M' such that M' and M are diffeomorphic through an orientation preserving diffeomorphism commuting with the inclusions of the source and the target into M and M'. From now on we wil call *right* the orientation of $\mathbf{1}$ and *opposite* the one of $\overline{\mathbf{1}}$.

Remark 3.5. A theorem (see [8] - Section IX.3) states that: *any two oriented connected compact bidimensional smooth manifolds are diffeomorphic if and only if they have the same Euler characteristic, the same number of right oriented component and the same number of opposite oriented components.*

Let $m, n \in \mathbb{N}$ and let M, M' be two oriented connected compact bidimensional smooth manifolds whose boundaries are $\overline{\boldsymbol{m}} \sqcup \boldsymbol{n}$. Then, by this theorem, M and M' induce the same morphism $\boldsymbol{m} \to \boldsymbol{n}$ of $\mathcal{S}$ if and only if they have the same Euler characteristic. As M and N are oriented, they are homeomorphic to connected sums of toruses and hence their Euler characteristics are $2 - 2g(M) - m - n$ and $2 - 2g(N) - m - n$ respectively, where $g(M)$ and $g(N)$ are their genuses, that is, their numbers of holes. Hence M and M' induce the same morphism $\boldsymbol{m} \to \boldsymbol{n}$ of $\mathcal{S}$ if and only if they have the same number of holes.

Proof of Theorem 3.4. At first let us prove that we are able to build representatives of the arrows of $\mathcal{S}$ *whose representatives are connected.* Let $m, n \in \mathbb{N}$ and let $[M] \colon \boldsymbol{m} \to \boldsymbol{n}$ be such that M is connected. Let $k := g(M)$. According to Remark 3.5, we only need to build an oriented connected compact bidimensional smooth manifold M' whose boundary is $\overline{\boldsymbol{m}} \sqcup \boldsymbol{n}$ (that is, a connected oriented cobordism form $\boldsymbol{m}$ to $\boldsymbol{n}$) and whose number of holes is k.

Firstly we build a connected oriented cobordism A from $\boldsymbol{m}$ to $\mathbf{1}$. If $m = 0$ i.e. $\boldsymbol{m} = \mathbf{0} = \emptyset$, then $A := lE$. If $m > 0$, then we parallel and glue (finitely many) copies of 1 and rF (if we consider their classes, this corresponds to applying the functor $\sqcup$ and composing respectively) in order to get a connected oriented cobordism A from $\boldsymbol{m}$ to $\mathbf{1}$. Analogously we build a connected oriented cobordism B from $\mathbf{1}$ to $\boldsymbol{n}$. If $n = 0$, then $B := rE$, otherwise we parallel and glue copies of 1 and lF.

If $k = 0$ we define $M' := BA$. This is a connected oriented cobordism from $\boldsymbol{m}$ to $\boldsymbol{n}$ without holes. We are done. Otherwise, if $k > 0$, observe that $(rF)(lF)$ is a connected oriented cobordism from $\mathbf{1}$ to $\mathbf{1}$ whose genus is 1. In other words $(rF)(lF)$ has one hole. Let $C^1 := (rF)(lF)$ and, for every $h > 1$, let $C^h := (C^{h-1})((rF)(lF))$. Hence C^k is a connected cobordism from $\mathbf{1}$ to $\mathbf{1}$ whose number of holes is k. Hence BC^kA is a connected oriented cobordism from $\boldsymbol{m}$ to $\boldsymbol{n}$ whose number of holes is k. Again we are done.

Secondly let us generalise the previous argument to arrows of $\mathcal{S}$ *whose representatives are disconnected.* Let $t \in \mathbb{N}$ be such that $t > 1$ and let us assume that, for every arrow $[X]$ of $\mathcal{S}$, if X has $d \in \mathbb{N}$ connected components and $d < t$ then there is a representative of $[X]$

obtained by gluing and paralleling lE, rE, lF, rF, 1 and T for finitely many times. Let us assume that $[M]\colon \boldsymbol{m} \to \boldsymbol{n}$ is an arrow of $\mathcal{S}$ such that M has $t \in \mathbb{N}$ connected components. In order to conclude, we only need to prove that there is a representative of $[M]$ obtained by gluing and paralleling lE, rE, lF, rF, 1 and T for finitely many times. Let X and Y be such that $M = X \sqcup Y$. Of course it is not necessarily the case that $[M] = [X] \sqcup [Y]$, as it is not necessarily the case that $\partial M_A = \partial X_A \sqcup \partial Y_A$ and $\partial M_B = \partial X_B \sqcup \partial Y_B$ (remind indeed that the order matters!). For instance it is not the case that $[T] = \tau_{(\mathbf{1},\mathbf{1})} = [1] \sqcup [1]$ even if $T = 1 \sqcup 1$. Anyway we observe that the equality $[T] \circ [T] = [1] \sqcup [1]$ holds, since $[T] \circ [T] = [TT] = (\mathbf{1} \sqcup \mathbf{1}) \times [0,1]$ and $(\mathbf{1} \sqcup \mathbf{1}) \times [0,1] = [1] \sqcup [1]$. The point is that T corresponds to a permutation of the cicumferences constituting $\mathbf{1} \sqcup \mathbf{1}$

A cobordism A from $\boldsymbol{m} \to \boldsymbol{m}$ obtained by gluing and paralleling 1 and T (for finitely many times) corresponds to a permutation of the m circumferences constituting the source of M. Analogously, we can consider a permutation of the n circumferences constituting the target of M by considering a cobordism B from $\boldsymbol{n} \to \boldsymbol{n}$ obtained by gluing and paralleling 1 and T (for finitely many times). Moreover, as 1 and T represent involutions of $\mathcal{S}$ ($[1]\circ[1] = 1_{\mathbf{1}}$ and $[T] \circ [T] = 1_{\mathbf{1}\sqcup\mathbf{1}}$), we can also construct representatives of the inverses of $[A]$ and $[B]$ by gluing and paralleling 1 and T (for finitely many times). Let A' be a representative of $[A]^{-1}$ obtained by gluing and paralleling lE, rE, lF, rF, 1 and T for finitely many times. Let B' be a representative of $[B]^{-1}$ obtained by gluing and paralleling lE, rE, lF, rF, 1 and T for finitely many times. Hence $[M] = [B'(BMA)A']$. Let $\tilde{X}$ and $\tilde{Y}$ be such that $BMA = \tilde{X} \sqcup \tilde{Y}$ and $\tilde{X} \supseteq X$ and $\tilde{Y} \supseteq Y$. We pick A and B in such a way that $[BMA] = [\tilde{X}] \sqcup [\tilde{Y}]$ (such a choice of A and B exists, as every permutation can be expressed through a cobordism as we saw before). Finally, as $\tilde{X}$ and $\tilde{Y}$ have less then t connected components (BMA has exactly t connected components), by inductive hypothesis it is the case that $[\tilde{X}]$ and $[\tilde{Y}]$ have representatives X' and Y' obtained by gluing and paralleling lE, rE, lF, rF, 1 and T for finitely many times. Hence it is the case that $[M] = [B'] \circ [BMA] \circ [A'] = [B'] \circ ([\tilde{X}] \sqcup [\tilde{Y}]) \circ [A'] = [B'] \circ ([X'] \sqcup [Y']) \circ [A']$ and then $B'(X' \sqcup Y')A'$ is a representative of $[M]$. As A', B', X', Y' are obtained by gluing and paralleling lE, rE, lF, rF, 1 and T for finitely many times, we are done. Q.E.D.

The class S of Theorem 3.4 is a generating class for $(\mathcal{S}, \sqcup, \emptyset, \tau)$. In order to get a presentation of $(\mathcal{S}, \sqcup, \emptyset, \tau)$ we also need a complete class of relations for S, which is provided by the following:

Theorem 3.6. *The following equalities between arrows of $\mathcal{S}$:*

$$
\begin{array}{lll}
[lE] \overset{\alpha_1}{=} [1] \circ [lE] & [1] \overset{\beta_1}{=} [1] \circ [1] & [rE] \overset{\gamma_1}{=} [rE] \circ [1] \\
[lF] \overset{\alpha_2}{=} [lF] \circ [1] & [T] \overset{\beta_2}{=} [T] \circ ([1] \sqcup [1]) & [rF] \overset{\gamma_2}{=} [1] \circ [rF] \\
[lF] \overset{\alpha_3}{=} ([1] \sqcup [1]) \circ [lF] & [T] \overset{\beta_3}{=} ([1] \sqcup [1]) \circ [T] & [rF] \overset{\gamma_3}{=} [rF] \circ ([1] \sqcup [1]) \\
[lF] \overset{\alpha_4}{=} [T] \circ [lF] & [1] \sqcup [1] \overset{\beta_4}{=} [T] \circ [T] & [rF] \overset{\gamma_4}{=} [rF] \circ [T]
\end{array}
$$

$$
\begin{array}{ll}
[1] \overset{\delta_1}{=} [rF] \circ ([lE] \sqcup [1]) & ([lF] \sqcup [1]) \circ [lF] \overset{\varepsilon_1}{=} ([1] \sqcup [lF]) \circ [lF] \\
[1] \overset{\delta_2}{=} [rF] \circ ([1] \sqcup [lE]) & [rF] \circ ([rF] \sqcup [1]) \overset{\varepsilon_2}{=} [rF] \circ ([1] \sqcup [rF]) \\
[1] \overset{\delta_3}{=} ([rE] \sqcup [1]) \circ [lF] & [lF] \circ [rF] \overset{\varepsilon_3}{=} ([1] \sqcup [rF]) \circ ([lF] \sqcup [1]) \\
[1] \overset{\delta_4}{=} ([1] \sqcup [rE]) \circ [lF] & [lF] \circ [rF] \overset{\varepsilon_4}{=} ([rF] \sqcup [1]) \circ ([1] \sqcup [lF])
\end{array}
$$

$$[T]\circ([lE]\sqcup[1])\overset{\zeta_1}{=}([1]\sqcup[lE])$$
$$([rE]\sqcup[1])\circ[T]\overset{\zeta_2}{=}([1]\sqcup[rE])$$
$$([rF]\sqcup[1])\circ([1]\sqcup[T])\circ([T]\sqcup[1])\overset{\zeta_3}{=}[T]\circ([1]\sqcup[rF])$$
$$([T]\sqcup[1])\circ([1]\sqcup[T])\circ([lF]\sqcup[1])\overset{\zeta_4}{=}([1]\sqcup[lF])\circ[T]$$
$$([T]\sqcup[1])\circ([1]\sqcup[T])\circ([T]\sqcup[1])\overset{\zeta_5}{=}([1]\sqcup[T])\circ([T]\sqcup[1])\circ([1]\sqcup[T])$$

hold. Moreover, the class R whose elements are these equalities is a complete class of relations for S.

Remark 3.7. Before the proof of Theorem 3.6 let us observe that, combining ζ_3 and β_4, we get the relation $([1]\sqcup[rF])\circ([T]\sqcup[1])\circ([1]\sqcup[T])\overset{\zeta_3'}{=}[T]\circ([rF]\sqcup[1])$. In fact:

$$\begin{aligned}([1]\sqcup[rF])\circ([T]\sqcup[1])\circ([1]\sqcup[T]) &= (([1]\circ[1])\sqcup([1]\circ[rF]))\circ([T]\sqcup[1])\circ([1]\sqcup[T])\\ &= ([1]\sqcup[1])\circ([1]\sqcup[rF])\circ([T]\sqcup[1])\circ([1]\sqcup[T])\\ &= [T]\circ[T]\circ([1]\sqcup[rF])\circ([T]\sqcup[1])\circ([1]\sqcup[T])\\ &= [T]\circ([rF]\sqcup[1])\circ([1]\sqcup[T])\circ([T]\sqcup[1])^2\circ([1]\circ[T])\\ &= [T]\circ([rF]\sqcup[1])\circ([1]\sqcup[1]\sqcup[1])\\ &= [T]\circ([rF]\sqcup[1]),\end{aligned}$$

where, apart from the functoriality and the associativity of $\sqcup$ and the neutrality relations of $[1]$, we have just applied ζ_3 and β_4. Analogously, combining γ_4 and β_4, we get the relation $([1]\sqcup[T])\circ([T]\sqcup[1])\circ([1]\sqcup[lF])\overset{\zeta_4'}{=}([lF]\sqcup[1])\circ[T]$.

Proof of Theorem 3.6. In order to prove that all the equalities holds, for each one of them we notice that the given representative on the left and the given representative on the right are both connected, have the same source and the same target, and have the same genus (it is always 0). Hence they induce the same arrow of $\mathcal{S}$. Clearly these equalities are relations for S, since they only involve finitely many compositions and $\sqcup$-parallelings of elements of the class S.

Hence we are left to prove that the class R is complete. Let us assume that $[M]$ is an arrow $\boldsymbol{m}\to\boldsymbol{n}$ (for some $m,n\in\mathbb{N}$) of $\mathcal{S}$ and that $[M]$ is equal to a given finite iteration $\mathfrak{I}$ of compositions and $\sqcup$-parallelings of the arrows of S. We are going to prove that, by applying finitely many times the equalities of R, we are able to move from the given expression of $[M]$ (the one provided by $\mathfrak{I}$) to a precise *fundamental expression* of $[M]$ (that we will talk about later). If we manage to prove this *main fact*, then we are done, since this fact implies in particular that any two expressions of $[M]$ as finite iterations of compositions and $\sqcup$-parallelings of the elements of S are actually equal modulo the elements of R. Hence, being $[M]$ an arbitrary arrow of $\mathcal{S}$, we will have proven that any equality between two given expressions of an arrow of $\mathcal{S}$ as finite iterations of compositions and $\sqcup$-parallelings of the elements of S (that is, any relation for S) can be deduced from the elements of R. The proof of the main fact is made of three steps:

(i) Firstly we assume that M is connected and that the iteration $\mathfrak{I}$ does not involve the usage of the arrow $[T]$.

(ii) Secondly we generalise the argument to expressions $\mathfrak{I}$ that may also involve the usage of the arrow $[T]$. This step is easily accomplished by proving that we can eliminate the occurrences of $[T]$ through finitely many applications of the elements of R).

(iii) Finally we extend the argument to possibily disconnected manifolds M.

(i) The fundamental expression of $[M]$ is $[BC^kA]$, whose representative is the connected oriented cobordism BC^kA of the first part of the proof of Theorem 3.4, being k the number of holes of M, i.e. its genus. Hence we are going to prove that $[M] = [BC^kA]$ by only using the elements of R.

At first we move all the $[rF]$-pieces on the left (in order to form the A-part) as much as we can, by following this procedure, until we cannot do anything anymore: if on the left of a $[rF]$-piece there is a [1]-piece, then we apply in sequence γ_3 - β_1 - γ_3 - γ_2, so that the $[rF]$-piece is moved to the left of the [1]-piece; if on the left of a $[rF]$-piece there is a $[lE]$-piece, then we apply in sequence γ_3 - α_1 - δ_1 or γ_3 - α_1 - δ_2, so that we eliminate both the $[rF]$-piece and the $[lE]$-piece and we get a [1]-piece (observe that through this procedure we eliminate all the $[lE]$-pieces, as they can only be on the left of some $[rF]$, being M connected); if on the left of a $[rF]$-piece there is a $[lF]$-piece glued through only one circumference, then we apply in sequence: γ_3 - α_3 - β_1 - β_1 - γ_3 - ε_3 or γ_3 - α_3 - β_1 - β_1 - γ_3 - ε_4, so that the $[rF]$-piece is moved to the left of the $[lF]$-piece; if on the left of a $[rF]$-piece there is a $[lF]$-piece glued through both the circumferences or another $[rF]$-piece, then we do not do anything.

Secondly we move all the $[lF]$-pieces on the right (in order to form the B-part) as much as we can, by following this procedure, until we cannot do anything anymore: if on the right of a $[lF]$-piece there is a [1]-piece, then we apply in sequence α_3 - β_1 - α_3 - α_2, so that the $[lF]$-piece is moved to the right of the [1]-piece; if on the right of a $[lF]$-piece there is a $[rE]$-piece, then we apply in sequence α_3 - γ_1 - δ_3 or α_3 - γ_1 - δ_4, so that we eliminate both the $[lF]$-piece and the $[rE]$-piece and we get a [1]-piece (as before, observe that through this procedure we eliminate all the $[rE]$-pieces, as they can only be on the right of some $[lF]$, being M connected); if on the right of a $[lF]$-piece there is a $[rF]$-piece glued through only one circumference, then (as before) we apply in sequence: γ_3 - α_3 - β_1 - β_1 - γ_3 - ε_3 or γ_3 - α_3 - β_1 - β_1 - γ_3 - ε_4, so that the $[lF]$-piece is moved to the right of the $[rF]$-piece; if on the right of a $[lF]$-piece there is a $[rF]$-piece glued through both the circumferences or another $[lF]$-piece, then we do not do anything.

Observe that these two procedures only take us finitely many applications of the elements of R. In particular, they involve all of them but ε_1, ε_2, α_4, β_2, β_3, β_4, γ_4 and ζ_i for $i \in \{1, 2, 3, 4, 5\}$. As after every step we always get a representative of the same arrow of $\mathcal{S}$ (and so, in particular, a compact oriented cobordism diffeomorphic to M), it is the case that -even once the procedures are complete- the number of holes is always k and therefore in the middle we must have a compact oriented cobordism C' made of finitely many [1]-pieces and k-many $[rF] \circ [lF]$-pieces. Hence, up to applying α_1 and γ_1 finitely many times in order to eliminate the [1]-pieces between the k-many $[rF] \circ [lF]$-pieces, we can assume that $C' = C^k$. Moreover, up to applying ε_2, β_1 and γ_i for $i \in \{1, 2, 3\}$ finitely many times, we can assume that the compact oriented cobordism glued on the left of C' is A. Analogously, up to applying ε_1, β_1 and α_i for $i \in \{1, 2, 3\}$ finitely many times, we can assume that the compact oriented cobordism glued on the right of C' is B. Hence we conclude that the equality $[M] = [BC^kA]$ can actually be inferred by applying the elements of R finitely many times.

(ii) Observe that up to now we have not used the equalities α_4, β_2, β_3, β_4, γ_4 and ζ_i for $i \in \{1,2,3,4,5\}$ of R, since they involve the arrow $[T]$ and in (i) we assumed the iteration $\mathfrak{I}$ not to involve it. Now, let us assume that in the given expression of $[M]$ as a finite iteration of compositions and $\sqcup$-parallelings of elements of S there are exactly $t \in \mathbb{N}$ occurrences of the arrow $[T]$, being $t > 0$. Moreover, let us assume that, whenever an expression of an arrow of $\mathcal{S}$ through finitely many compositions and $\sqcup$-parallelings of elements of S contains $d \in \mathbb{N}$ occurrences of $[T]$ and $d < t$, then it is the case that we can eliminate all the occurrences of $[T]$ from this expression by applying the elements of R for finitely many times. We are going to prove that the same holds for $[M]$. Indeed, if we manage to prove this, then we are done: we will have proven that, for every expression of an arrow of $\mathcal{S}$ through finitely many compositions and $\sqcup$-parallelings of elements of S, it is the case that we can eliminate all the occurrences of $[T]$ from this expression by applying the elements of R for finitely many times. After that, we will be done by applying (i).

Let us consider a $[T]$-piece appering in the expression of $[M]$. Because of the inductive hypothesis, we are done if we manage to eliminate the given $[T]$-piece without adding any other one. Up to applying α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2 and γ_3 finitely many times, we can assume that all the pieces parallel with the given $[T]$-piece (if they exist) are just cilinders (observe that this procedure does not modify the number of occurrences of the arrow $[T]$ in the given expression of $[M]$ as a finite iteration of compositions and $\sqcup$-parallelings of elements of S). Hence $[T]$ with its parallel pieces (it they exist) constitutes an arrow $[C]$ $\boldsymbol{k} \to \boldsymbol{k}$ for some $k \in \mathbb{N} \setminus \{0,1\}$. Hence we get a decomposition of $[M]$ as $[B] \circ [C] \circ [A]$, being $[A]$ an arrow $\boldsymbol{m} \to \boldsymbol{k}$ and being $[B]$ an arrow $\boldsymbol{k} \to \boldsymbol{n}$. Observe that the expressions of $[A]$ and $[B]$ contain less then t occurrences of $[T]$.

If A is connected then, because of the inductive hypothesis and because of (i), we can assume that the expression of $[A]$ is the fundamental one. Then up to applying ε_1, β_1 and α_i for $i \in \{1,2,3\}$ finitely many times, we can assume that there is a $[lF]$-piece of the expression of $[A]$ that is glued on the left of the given $[T]$-piece through both the circumferences. By applying α_4, we eliminate the given $[T]$-piece. We are done.

If B is connected then, because of the inductive hypothesis and because of (i), we can assume that the expression of $[B]$ is the fundamental one. Then up to applying ε_2, β_1 and γ_i for $i \in \{1,2,3\}$ finitely many times, we can assume that there is a $[rF]$-piece of the expression of $[A]$ that is glued on the right of the given $[T]$-piece through both the circumferences. By applying γ_4, we eliminate the given $[T]$-piece. Again, we are done.

Therefore, let us assume that both A and B are disconnected. Let A_1 be the unique connected component of A such that the codomain of $[A_1]$ contains the lower circumference of the domain of the given $[T]$-piece. Let A_2 be the unique connected component of A such that the codomain of $[A_2]$ contains the upper circumference of the domain of the given $[T]$-piece. Let B_1 be the unique connected component of B such that the domain of $[B_1]$ contains the lower circumference of the codomain of the given $[T]$-piece. Let B_2 be the unique connected component of B such that the domain of $[B_2]$ contains the upper circumference of the codomain of the given $[T]$-piece. Observe that there must exist at least one between: a cylinder parallel to $[T]$ connecting $[A_1]$ and $[B_1]$; a cylinder parallel to $[T]$ connecting $[A_2]$ and $[B_2]$ (or both). Indeed, if this was not the case, then $[M]$ would be disconnected. Without loss of generality, let us assume that there is a cylinder $[1]$ parallel to $[T]$ connecting $[A_1]$ and $[B_1]$. As A_1 and B_1 are connected and the expressions of $[A_1]$ and $[B_1]$ contain less then t occurences of $[T]$, because of the inductive hypothesis and because of (i), we can assume that the expressions of $[A_1]$ and $[B_1]$ are the fundamental ones. Therefore, up to applying ε_1, ε_2, β_i, α_i and γ_i for $i \in \{1,2,3\}$ finitely many times, we can assume that there are a $[lF]$-piece and a $[1]'$-piece appering in $[A_1]$ and a $[rF]$-piece and a $[1]''$-piece appearing in $[B_1]$ such that $[T]$ fits into a morphism $([rF] \sqcup [1]'') \circ ([1] \sqcup [T]) \circ ([lF] \sqcup [1]')$ appearing in the expression of $[M]$. Hence we are done if we prove that we can eliminate $[T]$ from the expression $([rF] \sqcup [1]) \circ ([1] \sqcup [T]) \circ ([lF] \sqcup [1])$ by just applying the elements of R finitely many time. In fact:

$$\begin{aligned}
([rF] \sqcup [1]) \circ ([1] \sqcup [T]) \circ ([lF] \sqcup [1]) &= ([rF] \sqcup [1]) \circ ([1] \sqcup [T]) \circ (([T] \circ [lF]) \sqcup ([1] \circ [1])) \\
&= ([rF] \sqcup [1]) \circ ([1] \sqcup [T]) \circ ([T] \sqcup [1]) \circ ([lF] \sqcup [1]) \\
&= [T] \circ ([1] \sqcup [rF]) \circ ([lF] \circ [1]) \\
&= [T] \circ [lF] \circ [rF] \\
&= [lF] \circ [rF]
\end{aligned}$$

where: the first equality is a consequence of α_4 and β_1, the second one of the functoriality of $\sqcup$, the third one of ζ_3, the fourth one of ε_3 and the last one of α_4. Observe that a path involving ζ_4 instead of ζ_3 and γ_4 instead of α_4 is also possible. We are done.

If there was not a cylinder $[1]$ parallel to $[T]$ connecting $[A_1]$ and $[B_1]$, that is, there is a cylinder $[1]$ parallel to $[T]$ connecting $[A_2]$ and $[B_2]$, then during this last argument we are required to prove that $([1] \sqcup [rF]) \circ ([T] \sqcup [1]) \circ ([1] \sqcup [lF]) = ([lF] \circ [rF])$ by only applying the elements of R finitely many times. We just apply ζ_3' (see Remark 3.7) instead of ζ_3 and ε_4 instead of ε_3. Again, observe that a path involving ζ_4' instead of ζ_3' and γ_4 instead of α_4 is also possible. Again, we are done.

(iii) Observe that up to now we have not used the equalities ζ_1, ζ_2 and ζ_5. Let us consider a given expression of $[M]$ and the equality $[M] = [B'] \circ [BMA] \circ [A']$ of the end of the Proof of Theorem 3.4, being A and B the permutation cobordisms (only made of $[T]$-pieces and $[1]$-pieces). The equalities $[A] \circ [A'] = [1]$ and $[B'] \circ [B] = [1]$ can be deduced by only applying finitely many times the elements β_1, β_2, β_3, β_4 and γ_5 of R. Then, as $[BMA]$ (by induction) is expressed as a finite $\sqcup$-paralleling of a finite family $\mathfrak{F}$ of arrows of $\mathcal{S}$ whose representatives are connected, by (ii) it is the case that the expressions of everyone of these arrows can be moved to the respective fundamental ones through finitely many applications of the elements of R. Up to change the order of the elements of $\mathfrak{F}$ in the final expression of $[M]$ (and up to add/eliminate $[1]$-pieces), this *generalised fundamental expression* is unique. Moreover these changes only involve the permutation parts $[A]$, $[B]$, $[A']$ and $[B']$ and can be achieved by only applying the relations of R involving the arrow $[T]$. Since the given initial expression of $[M]$ is arbitrary, we conclude that we can always move it to the generalised fundamental expression through finitely many applications of the elements of R. Hence we are done. Q.E.D.

We conclude this section by summarising the results of Theorem 3.4 and Theorem 3.6 into the following:

Corollary 3.8 (*Monoidal presentation of* $(\mathcal{S}, \sqcup, \emptyset, \tau)$)**.** *The couple (S, R) is a monoidal presentation of the symmetric monoidal category $(\mathcal{S}, \sqcup, \emptyset, \tau)$, being $\mathcal{S}$ the skeleton of* 2Cob *spanned by the object class $\{\emptyset, \mathbf{1}, \mathbf{2}, ...\}$, being S the generating class for $(\mathcal{S}, \sqcup, \emptyset, \tau)$ of Theorem 3.4 and being R the complete class of relations for S of Theorem 3.6.*

4 Bidimensional TQFTs & Frobenius Algebras

From now on let $\mathbb{K}$ be a field. As anticipated, we would like to give an explicit description of the 2-dimensional topological quantum field theories w.r.t. $\mathbb{K}$. At first we need to specify which kind of "explicit description" we are looking for. Usually in mathematics the most desired one is a so-called *classification theorem.* Whenever we talk about a *classification* of a class $\mathfrak{A}$ of objects, we implicitly mean that we have a notion of morphism between them turning $\mathfrak{A}$ into a category. Hence to classify $\mathfrak{A}$ corresponds to saying which are the objects of $\mathfrak{A}$ up to the notion of isomorphism of its categorical structure (in detail this also corresponds to partitioning the class $\mathfrak{A}$ into the equivalence classes of the mutually isomorphic object and then picking a representative for every one of them).

For instance, when we talk about the classification of the 2-dimensional compact connected topological manifolds we are implicitly considering the natural notion of *continuous map* as the appropriate notion of arrow turning the class of the 2-dimensional compact connected topological manifolds into a category. Hence in this case, to classify the 2-dimensional compact connected topological manifolds means to list all the 2-dimensional compact connected topological manifolds up to homeomorphism (the corresponding notion of isomorphism), that is, to pick a representative for every equivalence class of mutually homeomorphic 2-dimensional compact connected topological manifolds. A choice of these representatives is the following: a sphere, a torus, a projective plane and all the finite connected sums of them.

Coming back to our main enviroment, we remind that we already have a notion of arrow between two given 2-dimensional TQFTs w.r.t. $\mathbb{K}$, as these are the objects of the category $2\text{TQFT}(\mathbb{K}) = \mathbb{K}\text{-linRep}(2\text{Cob}, \sqcup, \emptyset, \tau)$, whose arrows are the monoidal natural transformations between them. Hence instinctively our aim should be to classify the 2-dimensional TQFTs up to monoidal natural isomorphism.

Remark 4.1 (*A classification is nothing but a skeleton [and an equivalence of categories is nothing but a "generalised classification"]*)**.** Observe that the classification of a given class of objects up to a given notion of arrow between these objects is nothing but the exhibition of a *skeleton* (see Appendix **3.**) of the corresponding categorical structure. We remind that a skeleton of a given category $\mathcal{C}$ is a full subcategory $\mathcal{S}$ of $\mathcal{C}$ such that the inclusion functor $\mathcal{S} \hookrightarrow \mathcal{C}$ is essentially surjective (hence an equivalence of categories) and such that every two isomorphic object of $\mathcal{S}$ are equal. As mentioned, to prove a classification theorem for a given category is equivalent to exhibiting a skeleton of that category. In other words, if $\mathfrak{A}$ is a category, then a subclass $\mathfrak{B}$ of $\mathfrak{A}$ constitutes a classification of $\mathfrak{A}$ precisely when the full subcategory of $\mathfrak{A}$ spanned by $\mathfrak{B}$ is a skeleton of $\mathfrak{A}$. For instance if $\mathfrak{A}$ is the category of the 2-dimensional compact connected topological manifolds and the continuous maps between them, then the full subcategory of $\mathfrak{A}$ spanned by a sphere, a torus, a projective plane and all their finite connected sums constitutes a skeleton of $\mathfrak{A}$. Another instance was given in the previous section: the full subcategory $\mathcal{S}$ of 2Cob spanned by $\emptyset$, a given circumference $\mathbf{1}$ and the finite disjoint unions of copies of $\mathbf{1}$ is a skeleton of 2Cob. Hence this also means that the set $\{\emptyset, \mathbf{1}, \mathbf{2}, ...\}$ classifies 2Cob.

In Remark 4.1 we observed that a classification of a given category $\mathfrak{A}$ is essentially an equivalence of categories $\mathfrak{A} \simeq \mathfrak{B}$ such that $\mathfrak{B}$ is a skeletal category, that is, for every two objects B, C of $\mathfrak{B}$, if $B \cong C$ then $B = C$. However, in our research of an "explicit description" of $2\text{TQFT}(\mathbb{K})$ we may also content ourselves with determining a generic equivalence of categories $2\text{TQFT}(\mathbb{K}) \simeq \mathfrak{B}$, without necessarily requiring $\mathfrak{B}$ to be a skeletal category.

Of course we would like $\mathfrak{B}$ to be simpler then $2\textsc{Cob}(\mathbb{K})$, so that we have a more concrete intuition of the behaviour of the TQFTs. We start by studying how the individual TQFTs look like. The monoidal presentation (S, R) of their domain $(\mathcal{S}, \sqcup, \emptyset, \tau)$ of Corollary 3.8 helps us characterise them.

Let F be a 2-dimensional topological quantum field theory w.r.t. $\mathbb{K}$, that is, a symmetric monoidal functor $(2\textsc{Cob}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma)$. As F is a functor $2\textsc{Cob} \to \mathbb{K}\text{-}\textsc{Vect}$, it preserves the isomorphisms. Therefore we may also assume that F assumes the same value on isomorphic objects and arrows of $2\textsc{Cob}$. This corresponds to restricting F to the skeleton $\mathcal{S}$ of $2\textsc{Cob}$ (see Section 3). Hence let us assume that F is a symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma)$. Actually we will see that the description of this kind of functors is sufficient in order to describe the category $2\textsc{tQft}(\mathbb{K})$. As F is monoidal, it is the case that $F\emptyset = \mathbb{K}$. Let $V := F\mathbf{1}$, which is a finite dimensional $\mathbb{K}$-linear space by Proposition 2.12. As F is monoidal and as $\boldsymbol{n}$ is the disjoint union of n copies of $\mathbf{1}$, for every $n \in \mathbb{N}$, it is the case that $F\boldsymbol{n} = \otimes_{i=1}^{n} F\mathbf{1} = \otimes_{i=1}^{n} V$.

The class S of Theorem 3.4 is a generating class for $(\mathcal{S}, \sqcup, \emptyset, \tau)$. Then we only need to specify the values of F on the arrows represented by the elements of S. Let:

$$\begin{aligned}
(\alpha\colon V \to V) &:= F([1]\colon \mathbf{1} \to \mathbf{1}) \\
(\beta\colon V \otimes V \to V \otimes V) &:= F([T]\colon \mathbf{2} \to \mathbf{2}) \\
(f\colon \mathbb{K} \to V) &:= F([lE]\colon \emptyset \to \mathbf{1}) \\
(g\colon V \to \mathbb{K}) &:= F([rE]\colon \mathbf{1} \to \emptyset) \\
(h\colon V \to V \otimes V) &:= F([lF]\colon \mathbf{1} \to \mathbf{2}) \\
(i\colon V \otimes V \to V) &:= F([rF]\colon \mathbf{2} \to \mathbf{1}).
\end{aligned}$$

The class R of Theorem 3.6 is a complete class of relations for S, that is, it contains the whole information regarding the categorical and the symmetric monoidal structure of $(\mathcal{S}, \sqcup, \emptyset, \tau)$. The elements of R are relations only involving the arrows represented by the elements of S, compositions and disjoint unions of them. As F is a monoidal functor, it needs to preserve the composition and to convert the disjoint union of two arrows of $\mathcal{S}$ into the tensor product of two arrows of $\mathbb{K}$-$\textsc{Vect}$. Hence the elements of R become relations involving the $\mathbb{K}$-linear maps 1_V, $\sigma_{(V,V)}$, f, g, h and i, compositions and tensor products of them. The relations α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2 and γ_3 in $\mathcal{S}$ characterise that $[1]$ is the identity arrow $1_{\mathbf{1}}\colon \mathbf{1} \to \mathbf{1}$ and F preserves the identity arrows, as it is a functor. Hence it is the case that $\alpha = F[1] = F1_{\mathbf{1}} = 1_V$. Analogously, the relations β_4, ζ_1, ζ_2, ζ_3, ζ_4 and ζ_5 characterise that $[T]$ is $\tau_{(\mathbf{1},\mathbf{1})}$. As F is a monoidal functor, it is the case that $F[T] = F\tau_{(\mathbf{1},\mathbf{1})} = \sigma_{(V,V)}$ i.e. $\beta = \sigma_{(V,V)}$. Hence we are left with the relations α_4, γ_4, δ_1, δ_2, δ_3, δ_4, ε_1, ε_2, ε_3 and ε_4. For instance, applying F to the members of ε_4, we get the relation

$F[lF] \circ F[rF] = (F[rF] \otimes F[1]) \circ (F[1] \otimes F[lF])$ i.e. $h \circ i = (i \otimes 1_V) \circ (1_V \otimes h)$. Analogously:

$$
\begin{array}{ll}
h \overset{(a)}{=} \sigma_{(V,V)} \circ h & i \overset{(a')}{=} i \circ \sigma_{(V,V)} \\
1_V \overset{(b)}{=} i \circ (f \otimes 1_V) & (h \otimes 1_V) \circ h \overset{(d)}{=} (1_V \otimes h) \circ h \\
1_V \overset{(b')}{=} i \circ (1_V \otimes f) & i \circ (i \otimes 1_V) \overset{(d')}{=} i \circ (1_V \otimes i) \\
1_V \overset{(c)}{=} (g \otimes 1_V) \circ h & h \circ i \overset{(e)}{=} (1_V \otimes i) \circ (h \otimes 1_V) \\
1_V \overset{(c')}{=} (1_V \otimes g) \circ h & h \circ i \overset{(e')}{=} (i \otimes 1_V) \circ (1_V \otimes h)
\end{array}
$$

are the relations on the arrows f, g, h and i provided by α_4, γ_4, δ_1, δ_2, δ_3, δ_4, ε_1, ε_2, ε_3 and ε_4. Let Φ be the set whose elements are these relations. We proved the following:

Proposition 4.2. *Let F be a symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$. Let $V := F\mathbf{1}$, $f := F[lE]$, $g := F[rE]$, $h := F[lF]$ and $i := F[rF]$. Then V is a finite dimensional $\mathbb{K}$-linear space and the 5-tuple (V, f, g, h, i) satisfies the elements of Φ.*

Hence every symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$ defines a particular structure in the symmetric monoidal category $(\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$. We will recognise this structure again at the ending part of the following subsection.

Frobenius algebras in $\mathbb{K}$-Vect

We remind that a $\mathbb{K}$-*algebra* is a couple (V, b) where V is a $\mathbb{K}$-vector space and b is a $\mathbb{K}$-bilinear map $V \times V \to V$. Moreover we say that (V, b) is a *unital associative* $\mathbb{K}$-*algebra* when b is associative and has a neutral element. Finally we remind that φ is a *homomorphism of unital associative* $\mathbb{K}$-*algebras* $(V, b) \to (V', b')$ if φ is a $\mathbb{K}$-linear map $V \to V'$ such that $\varphi \circ b = b' \circ (\varphi \times \varphi)$ and φ preserves the neutral element. A characterisation of these notions is provided by the following:

Proposition 4.3 (*"How to talk about $\mathbb{K}$-algebras by only involving the symmetric monoidal categorical structure of $\mathbb{K}$-VECT"*)**.** *A unital associative $\mathbb{K}$-algebra is precisely a triple (V, f, i) where V is a $\mathbb{K}$-linear space and f and i are $\mathbb{K}$-linear maps $\mathbb{K} \to V$ and $V \otimes V \to V$ respectively such that the relations (b), (b') and (d') of the set Φ are satisfied.*

Moreover, a homomorphism of unital associative $\mathbb{K}$-algebras $(V, f, i) \to (V', f', i')$ is precisely a $\mathbb{K}$-linear map $\varphi \colon V \to V'$ such that $\varphi \circ i = i' \circ (\varphi \otimes \varphi)$ and $\varphi \circ f = f'$.

Proof. Let V be a $\mathbb{K}$-linear space and let us assume that there is a $\mathbb{K}$-bilinear associative map $b \colon V \times V \to V$ with neutral element e. By the universal property of the couple $(V \otimes V, \otimes \colon V \times V \to V \otimes V)$ (see Appendix **2.**), there is unique an arrow $i \colon V \otimes V \to V$ of $\mathbb{K}$-VECT such that $i \circ \otimes = b$. The associativity of b directly implies the property (d'), since:

$$
\begin{aligned}
(i \circ (i \otimes 1_V))(x \otimes y \otimes z) &= i(i(x \otimes y) \otimes z) \\
[i \circ \otimes = b] &= b(b(x, y), z) \\
[\text{associativity of } b] &= b(x, b(y, z)) \\
[i \circ \otimes = b] &= i(x \otimes i(y \otimes z)) \\
&= (i \circ (1_V \otimes i))(x \otimes y \otimes z).
\end{aligned}
$$

for every $x, y, z \in V$. Now, let f be the $\mathbb{K}$-linear map $\mathbb{K} \to V$ such that $f(\lambda) = \lambda e$ for every $\lambda \in \mathbb{K}$. The neutrality of e w.r.t. b directly implies the property (b), since:

$$\begin{aligned}
(i \circ (f \otimes 1_V))(x) &= i((f \otimes 1_V)(x)) \\
[V = \mathbb{K} \otimes V] &= i((f \otimes 1_V)(1 \otimes x)) = i(f(1) \otimes x) \\
&= i(e \otimes x) \\
[i \circ \otimes = b] &= b(e, x) = x \\
&= 1_V(x)
\end{aligned}$$

for every $x \in V$, being 1 the neutral element of $\mathbb{K}$ w.r.t. its multiplication. Analogously, the neutrality of w.r.t. b directly implies the property (b').

Viceversa, if there exist $\mathbb{K}$-linear maps f and i as in the statement, one defines $b := (V \times V \xrightarrow{\otimes} V \otimes V \xrightarrow{i} V)$ (hence b is bilinear) and $e := f(1)$ and directly verifies that the property (d') implies the associativity of b and that the properties (b) and (b') imply the neutrality of e w.r.t. b.

Let φ be a $\mathbb{K}$-linear map $V \to V'$. If it is the case that $\varphi \circ b = b' \circ (\varphi \times \varphi)$, then $(\varphi \circ i)(x \otimes y) = (\varphi \circ b)(x, y) = (b' \circ (\varphi \times \varphi))(x, y) = b'(\varphi(x), \varphi(y)) = i'(\varphi(x) \otimes \varphi(y)) = (i' \circ (\varphi \otimes \varphi))(x \otimes y)$ for every $x, y \in V$, hence $\varphi \circ i = i' \circ (\varphi \otimes \varphi)$. Moreover, if $\varphi(e) = e'$, then $(\varphi \circ f)(\lambda) = \varphi(\lambda e) = \lambda \varphi(e) = \lambda e' = f'(\lambda)$ for every $\lambda \in \mathbb{K}$, that is $\varphi \circ f = f'$. Viceversa, if $\varphi \circ i = i' \circ (\varphi \otimes \varphi)$ and $\varphi \circ f = f'$ then one precomposes the members of the first equality with $\otimes$ and computes the members of the second one on 1, to get directly the equalities $\varphi \circ b = b' \circ (\varphi \times \varphi)$ and $\varphi(e) = e'$. We are done. Q.E.D.

This characterisation of the notion of unital associative $\mathbb{K}$-algebra is useful because it does not involve the obscure notion of *bilinearity*: the maps f and i are just $\mathbb{K}$-linear, that is, arrows of $\mathbb{K}$-Vect. From a category-theoretic point of view, this is one of the most important features enjoyed by the tensor product (see Appendix **2.**): it converts $\mathbb{K}$-multilinear maps into $\mathbb{K}$-linear ones. Hence the notion of unital associative $\mathbb{K}$-algebra can be fully discussed through the "language" of the symmetric monoidal category $(\mathbb{K}\text{-Vect}, \otimes, \mathbb{K}, \sigma)$.

Remark 4.4. Let (V, f, i) be a $\mathbb{K}$-algebra.

Let g be a $\mathbb{K}$-linear map $V \to \mathbb{K}$. Let η be the unique $\mathbb{K}$-linear map $V \otimes V \to \mathbb{K}$ such that $\eta(x \otimes y) = g(i(x \otimes y))$ for every $x, y \in V$. Then η is *associative* i.e. it is the case that $\eta(i(x \otimes y) \otimes z) = \eta(x \otimes i(y \otimes z))$ for every $x, y, z \in V$. Indeed, if $x, y, z \in V$, then:

$$\eta(i(x \otimes y) \otimes z) = g(i(i(x \otimes y) \otimes z)) \overset{(d)}{=} g(i(x \otimes i(y \otimes z))) = \eta(x \otimes i(y \otimes z)).$$

Viceversa, if η is an associative $\mathbb{K}$-linear map $V \otimes V \to \mathbb{K}$, that is, it holds that $\eta(i(x \otimes y) \otimes z) = \eta(x \otimes i(y \otimes z))$ for every $x, y, z \in V$, then there is a $\mathbb{K}$-linear map $g\colon V \to \mathbb{K}$ such that $g(x) = \eta(f(1) \otimes x) = \eta(x \otimes f(1))$ for every $x \in V$. Indeed, for every $x, y \in V$ it is the case that:

$$\eta(f(1) \otimes x) \overset{(b')}{=} \eta(f(1) \otimes i(x \otimes f(1))) = \eta(i(f(1) \otimes x) \otimes f(1)) \overset{(b)}{=} \eta(x \otimes f(1)).$$

These assignments $g \mapsto \eta$ and $\eta \mapsto g$ define a bijective correspondence between the set of the $\mathbb{K}$-linear maps $V \to \mathbb{K}$ and the set of the associative $\mathbb{K}$-linear maps $V \otimes V \to \mathbb{K}$. Indeed, let us assume that $g \mapsto \eta$ and that $\eta \mapsto g'$. Then:

$$g'(x) = \eta(f(1) \otimes x) = g(i(f(1) \otimes x)) \overset{(b)}{=} g(x)$$

for every $x \in V$, that is, $g = g'$. Viceversa, if $\eta \mapsto g$ and $g \mapsto \eta'$, then:

$$\eta'(x \otimes y) = g(i(x \otimes y)) = \eta(f(1) \otimes i(x \otimes y)) = \eta(i(f(1) \otimes x) \otimes y) \overset{(b)}{=} \eta(x \otimes y)$$

for every $x, y \in V$, that is, $\eta = \eta'$.

In order to prove Proposition 4.6 we need the following well-known:

Lemma 4.5. *Let us consider the functor* $(^*)\colon \mathbb{K}\text{-}\textsc{finVect} \to \mathbb{K}\text{-}\textsc{finVect}$ *such that:*

$$(X \xrightarrow{\alpha} Y) \mapsto (Y^* \xrightarrow{\alpha^*} X^*)$$

for every $\mathbb{K}$*-linear map* α*, being* α^* *the precomposition with* α*. Then* $(^*)$ *is an anti-equivalence of categories.*

Proof. We prove that $(^*)$ is its own pseudo-inverse, that is, that $(^*) \circ (^*) = 1_{\mathbb{K}\text{-}\textsc{finVect}}$. Let X be a finite dimensional $\mathbb{K}$-linear space. Let F_X be the $\mathbb{K}$-linear map $X \to X^{**}$ such that, for every $x \in X$, it is the case that $F_X(x)$ is the $\mathbb{K}$-linear map $X^* \to \mathbb{K}$ such that $(F_X(x))(f) = f(x)$ for every $f \in X^*$. Let us assume that $F_X(x)$ is the null element of X^{**} for some $x \in X$. Then $f(x) = (F_X(x))(f) = 0$ for every $f \in X^*$. In particular, if we consider a basis $\mathcal{B}$ of X, as the projections on the elements of $\mathcal{B}$ are elements of X^*, it is the case that all the projections of x on the elements of $\mathcal{B}$ are zero i.e. x is the null linear combination of the elements of $\mathcal{B}$ i.e. x is the null element of X. Then it is the case that F_X is injective and hence an isomorphism of $\mathbb{K}$-linear spaces by the rank-nullity theorem (remind that X^* has the same dimension of X because X is a finite dimensional vector space; in particular X^{**} has the same dimension of X as well). Observe that X is an arbitrary object of $\mathbb{K}$-$\textsc{finVect}$. Moreover, it is the case that the following diagram:

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "\alpha"] & X^{**} \arrow[d, "\alpha^{**}"] \\ Y \arrow[r, "F_Y"] & Y^{**} \end{tikzcd}$$

commutes for every arrow $\alpha\colon X \to Y$ of $\mathbb{K}$-$\textsc{finVect}$. Indeed, if $x \in X$ and $f \in Y^*$, then:

$$\begin{aligned} ((\alpha^{**} \circ F_X)(x))(f) &= (\alpha^{**}(F_X(x)))(f) = (F_X(x) \circ \alpha^*)(f) \\ &= (F_X(x))(\alpha^*(f)) = (F_X(x))(f \circ \alpha) \\ &= f(\alpha(x)) = (F_Y(\alpha(x)))(f) \\ &= ((F_Y \circ \alpha)(x))(f) \end{aligned}$$

i.e. $(\alpha^{**} \circ F_X)(x) = (F_Y \circ \alpha)(x)$, as $f \in Y^*$ is arbitrary, i.e. $\alpha^{**} \circ F_X = F_Y \circ \alpha$, as $x \in X$ is arbitrary. Hence F is a natural isomorphism $1_{\mathbb{K}\text{-}\textsc{finVect}} \to (^*) \circ (^*)$ and we are done. Q.E.D.

Proposition 4.6. *Let* (V, f, i) *be a unital associative* $\mathbb{K}$*-algebra and let us assume that* V *is finite dimensional. Let* g *be a* $\mathbb{K}$*-linear map* $V \to \mathbb{K}$ *and let* η *be the corresponding associative* $\mathbb{K}$*-linear map* $V \otimes V \to \mathbb{K}$ *(see Remark 4.4). Then the following are equivalent:*

(i) For every $y \in V \setminus \{0\}$ *there is* $x \in V$ *such that* $g(i(x \otimes y)) \neq 0$*.*

(ii) For every $x \in V \setminus \{0\}$ *there is* $y \in V$ *such that* $g(i(x \otimes y)) \neq 0$*.*

(iii) For every $y \in V \setminus \{0\}$ there is $x \in V$ such that $\eta(x \otimes y) \neq 0$.

(iv) For every $x \in V \setminus \{0\}$ there is $y \in V$ such that $\eta(x \otimes y) \neq 0$.

Proof. The equivalences $(i) \iff (iii)$ and $(ii) \iff (iv)$ are immediate, as for every $x, y \in V$ it is the case that $\eta(x \otimes y) = g(i(x \otimes y))$. Hence we are done if we prove that $(iii) \iff (iv)$.

Let us consider the $\mathbb{K}$-linear map $\eta_l \colon V \to V^*$ such that, for every $x \in V$, it is the case that $\eta_l(x)$ is the $\mathbb{K}$-linear map $V \to \mathbb{K}$ such that $(\eta_l(x))(y) = \eta(x \otimes y)$ for every $y \in V$. Analogously, let η_r be the $\mathbb{K}$-linear map $V \to V^*$ such that, for every $y \in V$, it is the case that $\eta_r(y)$ is the $\mathbb{K}$-linear map $V \to \mathbb{K}$ such that $(\eta_r(y))(x) = \eta(x \otimes y)$ for every $x \in V$. Observe that the statement (iii) is precisely the injectivity of η_r, because (iii) says that $\eta_r(y) \in V^*$ is nonzero whenever $y \in V$ is nonzero. Analogously (iv) is the injectivity of η_l. As V is finite dimensional, it is the case that V^* is finite dimensional as well and its dimension is the one of V. Hence $\eta_l, \eta_r \colon V \to V^*$ are injective if and only is they are isomorphisms of linear spaces, by the rank-nullity theorem. Therefore (iii) holds if and only if η_r is an isomorphism and (iv) holds if and only if η_l is an isomorphism. Then we are done it we prove that η_r is an isomorphism if and only if η_l is an isomorphism.

Let us consider the functor $(^*) \colon \mathbb{K}\text{-}\textsc{finVect} \to \mathbb{K}\text{-}\textsc{finVect}$ of Lemma 4.5. Since $(^*)$ is an equivalence of categories (as $(^*) \circ (^*) \cong 1_{\mathbb{K}\text{-}\textsc{finVect}}$), it is the case that η_l is an isomorphism if and only if η_l^* is an isomorphism. Hence we are done if we prove that η_r is an isomorphism if and only if η_l^* is an isomorphism. The following diagram:

$$\begin{tikzcd} V \arrow[r, "\eta_r"] \arrow[d, "F_V"'] & V^* \\ V^{**} \arrow[ur, "\eta_l^*"'] & \end{tikzcd}$$

commutes, being F_V the V-component of the natural isomorphism F of Lemma 4.5 (in particular F_V is an isomorphism of $\mathbb{K}$-linear spaces). Indeed, whenever $x, y \in V$, it is the case that:

$$\begin{aligned} ((\eta_l^* \circ F_V)(y))(x) &= (\eta_l^*(F_V(y)))(x) = (F_V(y) \circ \eta_l)(x) \\ &= (F_V(y))(\eta_l(x)) = (\eta_l(x))(y) \\ &= \eta(x \otimes y) = (\eta_r(y))(x) \end{aligned}$$

i.e. $(\eta_l^* \circ F_V)(y) = \eta_r(y)$, being $x \in V$ arbitrary, i.e. $\eta_l^* \circ F_V = \eta_r$, being $y \in V$ arbitrary. Hence, being F_V an isomorphism of $\mathbb{K}$-linear spaces, it is the case that η_r is an isomorphism of $\mathbb{K}$-linear spaces if and only if η_l^* is an isomorphism of $\mathbb{K}$-linear spaces. We are done. Q.E.D.

We are ready to give the fundamental:

Definition 4.7 (*Frobenius $\mathbb{K}$-algebra*)**.** Let (V, f, i) be a unital associative $\mathbb{K}$-algebra and let us assume that V is finite dimensional. A *Frobenius $\mathbb{K}$-algebra* is a 4-tuple (V, f, g, i), where $g \colon V \to \mathbb{K}$ is a $\mathbb{K}$-linear map satisfying (i) (and (ii)) of Proposition 4.6, or equivalently a 4-tuple (V, f, i, η), where $\eta \colon V \otimes V \to \mathbb{K}$ is an associative $\mathbb{K}$-linear map satisfying (iii) (and (iv)) of Proposition 4.6.

Now our aim is to characterise the notion of (commutative) Frobenius algebra in $\mathbb{K}$-Vect in such a way that we can talk about it by only involving the notions of $\mathbb{K}$-linear space, $\mathbb{K}$-linear map, composition in $\mathbb{K}$-Vect and tensor product of $\mathbb{K}$-linear spaces. I.e. by only involving the (symmetric) monoidal structure of $(\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma)$. We do this for two reasons: in order to prove that the commutative Frobenius $\mathbb{K}$-algebras are precisely the 2-dimensional TQFTs w.r.t $\mathbb{K}$; in order to be able to generalise the notion of (commutative) Frobenius algebra to an arbitrary symmetric monoidal category.

We remind that a unital associative $\mathbb{K}$-algebra (V, b, e) is *commutative* if the bilinear map $b\colon V \times V \to V$ is *symmetric* i.e. $b(x, y) = b(y, x)$ for every $x, y \in V$. If we consider the characterisation of the notion of unital associative $\mathbb{K}$-algebra of Proposition 4.3, that is, a triple (V, f, i) satisfying the elements (b), (b') and (d) of the set Φ, then (V, f, i) is commutative if and only if the property (a') is also satisfied. Indeed, if b is symmetric, then $i(x \otimes y) = b(x, y) = b(y, x) = i(y \otimes x)$ for every $x, y \in V$ i.e. $i(x \otimes y) = (i \circ \sigma_{(V,V)})(x \otimes y)$ for every $x, y \in V$. Then (as usual) $i(v) = (i \circ \sigma_{(V,V)})(v)$ for every $v \in V$, that is, $i = i \circ \sigma_{(V,V)}$. Viceversa, if (a') holds then we get the symmetry of b by just precomposing both the members with $\otimes$.

Definition 4.8. Let (V, f, g, i) be a Frobenius $\mathbb{K}$-algebra. We say that it is *commutative* if it is a commutative unital associative $\mathbb{K}$-algebra (V, f, i), that is, if condition (a') of Φ is satisfied.

We also need to consider the dual version of the notion of unital associative $\mathbb{K}$-algebra in the category $\mathbb{K}$-Vect:

Definition 4.9. Let V be a $\mathbb{K}$-linear space. Let us assume that g and h are $\mathbb{K}$-linear maps $V \to \mathbb{K}$ and $V \to V \otimes V$ respectively such that the properties (c), (c') and (d) of Φ are satisfied. Then we say that the triple (V, g, h) is a *counital coassociative* $\mathbb{K}$*-coalgebra*. We say that it is *cocommutative* if (a) holds as well.

Moreover, we say that $\varphi\colon (V, g, h) \to (V', g', h')$ is a *homomorphism of counital coassociative* $\mathbb{K}$*-coalgebras* if φ is a $\mathbb{K}$-linear map $V \to V'$ such that $h' \circ \varphi = (\varphi \otimes \varphi) \circ h$ and $g' \circ \varphi = g$.

Lemma 4.10. *Let (V, f, i) be a unital associative $\mathbb{K}$-algebra. Let η be an associative $\mathbb{K}$-linear map $V \otimes V \to \mathbb{K}$. Then V is finite dimensional and the condition (iv) of Proposition 4.6 holds if and only if there is a $\mathbb{K}$-linear map $\gamma\colon \mathbb{K} \to V \otimes V$ such that:*

$$(V = V \otimes \mathbb{K} \xrightarrow{1_V \otimes \gamma} V \otimes V \otimes V \xrightarrow{\eta \otimes 1_V} \mathbb{K} \otimes V = V) = (V \xrightarrow{1_V} V).$$

Analogously, V is finite dimensional and the condition (iii) holds if and only if there is a $\mathbb{K}$-linear map $\gamma'\colon \mathbb{K} \to V \otimes V$ such that:

$$(V = \mathbb{K} \otimes V \xrightarrow{\gamma' \otimes 1_V} V \otimes V \otimes V \xrightarrow{1_V \otimes \eta} V \otimes \mathbb{K} = V) = (V \xrightarrow{1_V} V).$$

Proof. We prove the first statement. Let us assume that such a γ exists. Then, as in the proof of Proposition 2.12, there are $k \in \mathbb{N} \setminus \{0\}$ and $y_i, x_i \in V \setminus \{0\}$ for $i \in \{1, ..., k\}$ such that $\gamma(1) = \sum_{i=1}^{k}(y_i \otimes x_i)$, being 1 the neutral element of $\mathbb{K}$ w.r.t. its multiplication. Again, as in the proof of Proposition 2.12, for every $x \in V$ it is the case that $x \overset{\clubsuit}{=} \sum_{i=1}^{k} \eta(x \otimes y_i)x_i$ and hence V is finite dimensional. Let us assume that $\eta_l(x) \in V^*$ is the null element of V^* for some $x \in V$ (see the proof of Proposition 4.6). Then in particular for every $i \in \{1, ..., k\}$ it is the case that $\eta(x \otimes y_i) = (\eta_l(x))(y_i) = 0$. Then, according to the equality $\clubsuit$, it is the case that x is the null element of V. Hence η_l is injective, and this is equivalent to the condition (iii) of Proposition 4.6 (see the proof of Proposition 4.6).

Viceversa let us assume that V is finite dimensional and that (iii) holds. Hence there is a finite basis $\{x_i\}_{i=1}^{k}$ of V and η_l is injective. Being η_l injective, it is the case that $\{\eta_l(x_i)\}_{i=1}^{k}$

is also linearly independent and this implies that there is a set $\{y_i \in V \setminus \{0\}\}_{i=1}^k$ such that, for every $i, j \in \{1, ..., k\}$, it is the case that $(\eta_l(x_i))(y_j) = \delta_{i,j}$. Let γ be the unique $\mathbb{K}$-linear map $\mathbb{K} \to V \otimes V$ such that $\gamma(1) = \sum_{i=1}^k y_i \otimes x_i$, being 1 the neutral element of $\mathbb{K}$ w.r.t. its multiplication. It is the case that the equality $(V = V \otimes \mathbb{K} \xrightarrow{1_V \otimes \gamma} V \otimes V \otimes V \xrightarrow{\eta \otimes 1_V} \mathbb{K} \otimes V = V) = (V \xrightarrow{1_V} V)$ holds.

Analogously one can prove the second part of the statement. Q.E.D.

Remark 4.11. Let (V, f, i) be a unital associative $\mathbb{K}$-algebra and let η be an associative $\mathbb{K}$-linear map $V \otimes V \to \mathbb{K}$. By Proposition 4.6 it is the case that V is finite dimensional and (iv) holds if and only if V is finite dimensiona and (iii) holds. Hence there is a $\mathbb{K}$-linear map γ as in Lemma 4.10 if and only if there is a $\mathbb{K}$-linear map γ' as in Lemma 4.10. Moreover it is the case that γ and γ' coincide: observe that $\mathbb{K}$-linear map $(\mathbb{K} \xrightarrow{\gamma' \otimes \gamma} V \otimes V \otimes V \otimes V \xrightarrow{1_V \otimes \eta \otimes 1_V} V \otimes V)$ is both equal to $(\mathbb{K} \xrightarrow{\gamma'} V \otimes V \xrightarrow{1_V \otimes 1_V \otimes \gamma} V \otimes V \otimes V \otimes V \xrightarrow{1_V \otimes \eta \otimes 1_V} V \otimes V) = (\mathbb{K} \xrightarrow{\gamma'} V \otimes V \xrightarrow{1_V \otimes 1_V} V \otimes V) = (\mathbb{K} \xrightarrow{\gamma'} V \otimes V)$ and to $(\mathbb{K} \xrightarrow{\gamma} V \otimes V \xrightarrow{\gamma' \otimes 1_V \otimes 1_V} V \otimes V \otimes V \otimes V \xrightarrow{1_V \otimes \eta \otimes 1_V} V \otimes V) = (\mathbb{K} \xrightarrow{\gamma'} V \otimes V \xrightarrow{1_V \otimes 1_V} V \otimes V) = (\mathbb{K} \xrightarrow{\gamma'} V \otimes V)$ (remind that in our convention $\mathbb{K}$-VECT satisfies $\mathbb{K} \otimes V = V = V \otimes \mathbb{K}$).

Observe that this argument also implies that such a γ (or γ') is unique. Hence we can summarise what we know in the following:

Proposition 4.12. *Let (V, f, i) be a unital associative $\mathbb{K}$-algebra. Let g be a $\mathbb{K}$-linear map $V \to \mathbb{K}$ and let η be the corresponding associative $\mathbb{K}$-linear map $V \otimes V \to \mathbb{K}$ (see Remark 4.4). Then the following are equivalent:*

1. *The $\mathbb{K}$-linear space V is finite dimensional and (one of) the equivalent conditions (i), (ii), (iii) and (iv) of Proposition 4.6 hold(s).*

2. *There exists a $\mathbb{K}$-linear map $\gamma\colon \mathbb{K} \to V \otimes V$ such that (one of) the equivalent conditions $1_V = (\eta \otimes 1_V) \circ (1_V \otimes \gamma)$ and $1_V = (1_V \otimes \eta) \circ (\gamma \otimes 1_V)$ hold(s).*

3. *There exists unique a $\mathbb{K}$-linear map $\gamma\colon \mathbb{K} \to V \otimes V$ such that (one of) the equivalent conditions $1_V = (\eta \otimes 1_V) \circ (1_V \otimes \gamma)$ and $1_V = (1_V \otimes \eta) \circ (\gamma \otimes 1_V)$ hold(s).*

Proof. Combine Proposition 4.6, Lemma 4.10 and Remark 4.11. Q.E.D.

Corollary 4.13. *Let (V, f, i) be a unital associative $\mathbb{K}$-algebra. Let $\eta\colon V \otimes V \to \mathbb{K}$ be an associative $\mathbb{K}$-linear map and let g be the corresponding $\mathbb{K}$-linear map $V \to \mathbb{K}$. Then (V, f, i, η) is a Frobenius $\mathbb{K}$-algebra, that is, (V, f, g, i) is a Frobenius $\mathbb{K}$-algebra, if and only if 2. or 3. of Proposition 4.12 are satisfied.*

We have everything we need in order to prove the following fundamental:

Theorem 4.14. *Let (V, f, g, i) be a Frobenius $\mathbb{K}$-algebra. Then there is unique a $\mathbb{K}$-linear map $h\colon V \to V \otimes V$ such that (V, g, h) is a counital coassociative $\mathbb{K}$-coalgebra (that is, the realtions (c), (c') and (d) of Φ hold) and the relations (e) and (e') of Φ hold.*

Proof. Let us consider the corresponding associative $\mathbb{K}$-linear map $\eta\colon V \otimes V \to \mathbb{K}$ and let γ be the unique $\mathbb{K}$-linear map $\gamma\colon \mathbb{K} \to V \otimes V$ of point *3.* of Proposition 4.12.

At first, let $\varphi\colon V\otimes V\otimes V\to\mathbb{K}$ be the unique $\mathbb{K}$-linear map such that $\varphi(x\otimes y\otimes z)=\eta(i(x\otimes y)\otimes z)=\eta(x\otimes i(y\otimes z))$ for every $x,y,z\in V$. In other words, $\varphi=\eta\circ(i\otimes 1_V)=\eta\circ(1_V\otimes i)$. Let us observe that $(V\otimes V\xrightarrow{1_V\otimes 1_V\otimes\gamma}V\otimes V\otimes V\otimes V\xrightarrow{\varphi\otimes 1_V}V)=(V\otimes V\xrightarrow{i}V)$. Indeed:

$$\begin{aligned}(\varphi\otimes 1_V)\circ(1_V\otimes 1_V\otimes\gamma)&=(\eta\otimes 1_V)\circ(i\otimes 1_V\otimes 1_V)\circ(1_V\otimes 1_V\otimes\gamma)\\&=(\eta\otimes 1_V)\circ((i\circ(1_V\otimes 1_V))\otimes((1_V\otimes 1_V)\circ\gamma))\\&=(\eta\otimes 1_V)\circ(i\otimes\gamma)\\ [(\eta\otimes 1_V)\circ(1_V\otimes\gamma)=1_V]&=(\eta\otimes 1_V)\circ(1_V\otimes\gamma)\circ i\\&=1_V\circ i=i.\end{aligned}$$

Analogously it is the case that $(V\otimes V\xrightarrow{\gamma\otimes 1_V\otimes 1_V}V\otimes V\otimes V\otimes V\xrightarrow{1_V\otimes\varphi}V)=(V\otimes V\xrightarrow{i}V)$. Hence it is the case that $(V\xrightarrow{\gamma\otimes 1_V}V\otimes V\otimes V\xrightarrow{1_V\otimes i}V\otimes V)=(V\xrightarrow{1_V\otimes\gamma}V\otimes V\otimes V\xrightarrow{i\otimes 1_V}V\otimes V)$, as:

$$\begin{aligned}(1_V\otimes i)\circ(\gamma\otimes 1_V)&=(1_V\otimes\varphi\otimes 1_V)\circ(1_V\otimes 1_V\otimes 1_V\otimes\gamma)\circ(\gamma\otimes 1_V)\\&=(1_V\otimes\varphi\otimes 1_V)\circ(\gamma\otimes 1_V\otimes 1_V\otimes 1_V)\circ(1_V\otimes\gamma)\\&=(i\otimes 1_V)\circ(1_V\otimes\gamma).\end{aligned}$$

Finally, we define $(V\xrightarrow{h}V\otimes V):=(1_V\otimes i)\circ(\gamma\otimes 1_V)=(i\otimes 1_V)\circ(1_V\otimes\gamma)$. By applying the relations $1_V=(\eta\otimes 1_V)\circ(1_V\otimes\gamma)$ and $1_V=(1_V\otimes\eta)\circ(\gamma\otimes 1_V)$ we also get that the dual realtion $(\eta\otimes 1_V)\circ(1_V\otimes h)=i=(1_V\otimes\eta)\circ(h\otimes 1_V)$. Now we prove that (V,g,h) is a counital coassociative $\mathbb{K}$-coalgebra. The element (d) of Φ holds because:

$$\begin{aligned}(h\otimes 1_V)\circ h&=(1_V\otimes i\otimes 1_V)\circ(\gamma\otimes 1_V\otimes 1_V)\circ(i\otimes 1_V)\circ(1_V\otimes\gamma)\\&=(1_V\otimes i\otimes 1_V)\circ(\gamma\otimes i\otimes 1_V)\circ(1_V\otimes\gamma)\\&=(1_V\otimes i\otimes 1_V)\circ(1_V\otimes 1_V\otimes i\otimes 1_V)\circ(\gamma\otimes 1_V\otimes\gamma)\\ [(d')]&=(1_V\otimes i\otimes 1_V)\circ(1_V\otimes i\otimes 1_V\otimes 1_V)\circ(\gamma\otimes 1_V\otimes\gamma)\\&=(1_V\otimes i\otimes 1_V)\circ(1_V\otimes i\otimes\gamma)\circ(\gamma\otimes 1_V)\\&=(1_V\otimes i\otimes 1_V)\circ(1_V\otimes 1_V\otimes\gamma)\circ(1_V\otimes i)\circ(\gamma\otimes 1_V)\\&=(1_V\otimes h)\circ h.\end{aligned}$$

Moreover, the element (c) holds as well, since:

$$\begin{aligned}(g\otimes 1_V)\circ h&=((g\circ 1_V)\otimes 1_V)\circ h\\ [(b)]&=((g\circ i\circ(f\otimes 1_V))\otimes 1_V)\circ h\\&=((\eta\circ(f\otimes 1_V))\otimes 1_V)\circ h\\&=(\eta\otimes 1_V)\circ(f\otimes 1_V\otimes 1_V)\circ h\\&=(\eta\otimes 1_V)\circ(1_V\otimes h)\circ(f\otimes 1_V)\\ [(\eta\otimes 1_V)\circ(1_V\otimes h)=i]&=i\circ(f\otimes 1_V)\\ [(b)]&=1_V.\end{aligned}$$

Analogously, by applying the element (b') we get that the element (c') holds as well. Now we prove that (e) and (e') hold. Indeed:

$$\begin{aligned}
h \circ i &= (1_V \otimes i) \circ (\gamma \otimes 1_V) \circ i \\
&= (1_V \otimes i) \circ (1_V \otimes 1_V \otimes 1_V) \circ (\gamma \otimes i) \\
&= (1_V \otimes i) \circ (1_V \otimes 1_V \otimes i) \circ (\gamma \otimes 1_V \otimes 1_V) \\
[(d')] &= (1_V \otimes i) \circ (1_V \otimes i \otimes 1_V) \circ (\gamma \otimes 1_V \otimes 1_V) \\
[h = (1_V \otimes i) \circ (\gamma \otimes 1_V)] &= (1_V \otimes i) \circ (h \otimes 1_V)
\end{aligned}$$

and (e) holds. Analogously one proves that (e') holds as well.

We are left to prove that such an h is unique. Let h' be a $\mathbb{K}$-linear map $V \to V \otimes V$ such that the properties (d), (c), (c'), (e) and (e') hold with h' in place of h. *Observe that actually only* (c) *and* (e') *are needed.* Then:

$$\begin{aligned}
(\eta \otimes 1_V) \circ (1_V \otimes (h' \circ f)) &= ((g \circ i) \otimes 1_V) \circ (1_V \otimes (h' \circ f)) \\
&= (g \otimes 1_V) \circ (i \otimes 1_V) \circ (1_V \otimes h') \circ (1_V \otimes f) \\
[(e')] &= (g \otimes 1_V) \circ h' \circ i \circ (1_V \otimes f) \\
[(c), (b')] &= 1_V \circ 1_V = 1_V
\end{aligned}$$

that is, by Remark 4.11, it is the case that $h' \circ f = \gamma$. Hence observe that:

$$\begin{aligned}
h' &= h' \circ 1_V \\
[(b')] &= h' \circ i \circ (1_V \otimes f) \\
[(e')] &= (i \otimes 1_V) \circ (1_V \otimes h') \circ (1_V \otimes f) \\
&= (i \otimes 1_V) \circ (1_V \otimes (h' \circ f)) \\
[h' \circ f = \gamma] &= (i \otimes 1_V) \circ (1_V \otimes \gamma) \\
[\text{definition of } h] &= h.
\end{aligned}$$

We are done. Q.E.D.

The opposite result holds as well:

Theorem 4.15. *Let* (V, f, g, h, i) *be a 5-tuple such that* (V, f, i) *is a unital associative* $\mathbb{K}$-*algebra (that is, the relations* (b), (b') *and* (d') *of* Φ *hold) and* (V, g, h) *is a counital coassociative* $\mathbb{K}$-*coalgebra (that is, the relations* (c), (c') *and* (d) *of* Φ *hold). Moreover let us assume that the relations* (e) *and* (e') *of* Φ *hold as well. Then* (V, f, g, i) *is a Frobenius* $\mathbb{K}$-*algebra (and in particular* V *is finite dimensional).*

Proof. Let η be the associative $\mathbb{K}$-linear map $V \otimes V \to \mathbb{K}$ corresponding to g, that is $\eta = g \circ i$ (see Remark 4.4). Moreover, let $\gamma := h \circ f$. Then it is the case that:

$$\begin{aligned}
(\eta \otimes 1_V) \circ (1_V \otimes \gamma) &= (g \otimes 1_V) \circ (i \otimes 1_V) \circ (1_V \otimes h) \circ (1_V \otimes f) \\
[(e')] &= (g \otimes 1_V) \circ h \circ i \circ (1_V \otimes f) \\
[(c), (b')] &= 1_V \circ 1_V = 1_V.
\end{aligned}$$

Then condition *2.* of Proposition 4.12 is verified and by Corollary 4.13 it is the case that (V, f, g, i) is a Frobenius $\mathbb{K}$-algebra (and in particular V is finite dimensional). Q.E.D.

Corollary 4.16 (*"How to talk about Frobenius $\mathbb{K}$-algebras by only involving the symmetric monoidal categorical structure of $\mathbb{K}$-*VECT*"*)**.** *A Frobenius $\mathbb{K}$-algebra is precisely a 5-tuple (V, f, g, h, i) such that (V, f, i) is a unital associative $\mathbb{K}$-algebra and (V, g, h) is a counital coassociative $\mathbb{K}$-coalgebra and the relations (e) and (e') of Φ hold. In other words, a Frobenius $\mathbb{K}$-algebra is precisely a 5-tuple (V, f, g, h, i) such that the relation (b), (b'), (c), (c'), (d), (d'), (e) and (e') of Φ hold. In particular V is finite dimensional.*

Proof. Combine Theorem 4.14 and Theorem 4.15. Q.E.D.

We said that a Frobenius $\mathbb{K}$-algebra (V, f, g, i) is commutative if and only if the unital associative $\mathbb{K}$-algebra (V, f, i) is commutative, that is, if and only if condition (a') of Φ holds. Equivalently a Frobenius $\mathbb{K}$-algebra (V, f, g, h, i) is commutative if and only if (V, f, i) is commutative i.e. if and only if condition (a') holds. Analogously, we say that:

Definition 4.17. A Frobenius $\mathbb{K}$-algebra (V, f, g, h, i) is *cocommutative* if and only if the counital coassociative $\mathbb{K}$-algebra (V, g, h) is cocommutative i.e. if and only if condition (a) of Φ holds (see Definition 4.9).

Proposition 4.18. *Let (V, f, g, h, i) be a Frobenius $\mathbb{K}$-algebra. Then it is commutative if and only if it is cocommutative.*

Proof. We only need to prove the equivalence $(a') \iff (a)$. Let us prove the implication $(a') \implies (a)$ (the other one is analogous). Then let us assume that (a') holds and let us prove that (a) holds as well. Let $h' := \sigma_{(V,V)} \circ h \colon V \to V \otimes V$. Then we only need to prove that $h' = h$. As we saw in the proof of Theorem 4.14 (the proof of the uniqueness of h), this follows if h' verifies (c) and (e') in place of h. Hence it suffices to prove that h' verifies them. The property (c) for h' is just the property (c') for h, as $(g \otimes 1_V) \circ h' = (g \otimes 1_V) \circ \sigma_{(V,V)} \circ h = (1_V \otimes g) \circ h \overset{(c')}{=} 1_V$. Moreover:

$$\begin{aligned}
(i \otimes 1_V) \circ (1_V \otimes h') &= (i \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}) \circ (1_V \otimes h) \\
[(a')] &= ((i \circ \sigma_{(V,V)}) \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}) \circ (1_V \otimes 1_V \otimes 1_V) \circ (1_V \otimes h) \\
[(\sigma_{(V,V)})^2 = 1_V \otimes 1_V] &= (i \otimes 1_V) \circ (\sigma_{(V,V)} \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V)^2 \circ (1_V \circ h) \\
[\clubsuit] &= (i \otimes 1_V) \circ ((1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V))^2 \circ (1_V \circ h) \\
[\spadesuit] &= \sigma_{(V,V)} \circ (1_V \otimes i) \circ (h \otimes 1_V) \circ \sigma_{(V,V)} \\
[(e) \text{ for } h] &= \sigma_{(V,V)} \circ h \circ i \circ \sigma_{(V,V)} \\
[(a')] &= h' \circ i
\end{aligned}$$

where: the equality $\clubsuit$ holds because $((\sigma_{(V,V)} \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V))(x \otimes y \otimes z) = (z \otimes y \otimes x) = ((1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}))(x \otimes y \otimes z)$ for every $x, y, z \in V$, hence $(\sigma_{(V,V)} \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V) = (1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)})$; the equality $\spadesuit$ holds because $((i \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V))(x \otimes y \otimes z) = i(y \otimes z) \otimes x = (\sigma_{(V,V)} \circ (1_V \otimes i))(x \otimes y \otimes z)$ and $((1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V) \circ (1_V \circ h))(x \otimes y) = (h(y) \otimes x) = ((h \otimes 1_V) \circ \sigma_{(V,V)})(x \otimes y)$ for every $x, y, z \in V$, hence $(i \otimes 1_V) \circ (1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V) = \sigma_{(V,V)} \circ (1_V \otimes i)$ and $(1_V \otimes \sigma_{(V,V)}) \circ (\sigma_{(V,V)} \otimes 1_V) \circ (1_V \circ h) = (h \otimes 1_V) \circ \sigma_{(V,V)}$. *Observe that the equalities $\spadesuit$ and $\clubsuit$ are the analogous of the equalities ζ_5 and ζ_3, ζ_4 of Theorem 3.6 involving the arrow $[T]$ and verified characterising that $[T]$ is $\tau_{(\mathbf{1},\mathbf{1})}$.* We proved that the property (e') for h' holds and we are done. Q.E.D.

Corollary 4.19. *A commutative Frobenius $\mathbb{K}$-algebra is precisely a 5-tuple (V, f, g, h, i) such that all the relations of the set Φ hold. In particular V is finite dimensional.*

Corollary 4.20. *Let F be a symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{Vect}, \otimes, \mathbb{K}, \sigma)$. Let $V := F\mathbf{1}$, $f := F[lE]$, $g := F[rE]$, $h := F[lF]$ and $i := F[rF]$. Then the 5-tuple (V, f, g, h, i) is a commutative Frobenius $\mathbb{K}$-algebra. In particular V is finite dimensional.*

Proof. Combine Proposition 4.2 and Corollary 4.19. Q.E.D.

We proved that every symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{Vect}, \otimes, \mathbb{K}, \sigma)$ is in particular a commutative Frobenius $\mathbb{K}$-algebra. As anticipated, we would also like to prove that the commutative Frobenius $\mathbb{K}$-algebras are precisely the symmetric monoidal functors $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{Vect}, \otimes, \mathbb{K}, \sigma)$.

Remark 4.21. Let (V, f, g, h, i) be a commutative Frobenius $\mathbb{K}$-algebra. Let F be the unique symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{Vect}, \otimes, \mathbb{K}, \sigma)$ such that $F\mathbf{1} = V$, $F[lE] = f$, $F[rE] = g$, $F[lF] = h$ and $F[rF] = i$. As the objects of $\mathcal{S}$ are the finite disjoint unions of copies of $\mathbf{1}$ and S (see Theorem 3.4) is a generating class of the symmetric monoidal category $(\mathcal{S}, \sqcup, \emptyset, \tau)$, a symmetric monoidal functor whose domain is $(\mathcal{S}, \sqcup, \emptyset, \tau)$ is completely determined by the values that it assumes on the objects $\emptyset$, $\mathbf{1}$ and the arrows $[1]$, $[T]$, $[lE]$, $[rE]$, $[lF]$, $[rF]$. However, if a functor is symmetric monoidal then the values on $\emptyset$, $[1]$ and $[T]$ are forced to be $\mathbb{K}$, $1_\bullet$ and $\sigma_{(\bullet,\bullet)}$ respectively, being $\bullet$ the value assumed on $\mathbf{1}$. Hence, if F exists then it is unique. Let us verify that these assignments actually define a symmetric monoidal functor. As we want F to be a symmetric monoidal functor, in the definition of F we mean that $F\emptyset = \mathbb{K}$, $F[1] = 1_V$ and $F[T] = \sigma_{(V,V)}$. Then, if F preserves the compositions and converts the functor $\sqcup$ into the functor $\otimes$, it is the case that F automatically preserves the identities, converts the $\emptyset$-functor into the $\mathbb{K}$-functor and sends τ to σ. Therefore in order to conclude we only need to prove that F preserves the composition and that it converts the functor $\sqcup$ into the functor $\otimes$. As R (see Theorem 3.6) is a complete class of relations for S, every possible equality involving the elements of S, the composition and the $\sqcup$-relation only is a consequence of the elements of R. Hence, in order to prove that F preserves the composition and converts $\sqcup$ into $\otimes$ (i.e. in order to conclude) it suffices to verify that this happens for the elements of R i.e. that the elements of the set Φ hold. But this is indeed the case because (V, f, g, h, i) is a commutative Frobenius $\mathbb{K}$-algebra. We are done.

Corollary 4.22 (*Characterisation of the bidimensional TQFTs w.r.t. $\mathbb{K}$ (restricted to $\mathcal{S}$)*)**.** *A symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{Vect}, \otimes, \mathbb{K}, \sigma)$ is precisely a commutative Frobenius $\mathbb{K}$-algebra. That is, the image (V, f, g, h, i) of a given symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{Vect}, \otimes, \mathbb{K}, \sigma)$ is a commutative Frobenius $\mathbb{K}$-algebra and, whenever (V, f, g, h, i) is a commutative Frobenius $\mathbb{K}$-algebra, there is unique a symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\mathrm{Vect}, \otimes, \mathbb{K}, \sigma)$ whose image is (V, f, g, h, i).*

Proof. Combine Corollary 4.20 and Remark 4.21. Q.E.D.

Main result about the bidimensional TQFTs

We essentially proved that every bidimensional TQFT w.r.t $\mathbb{K}$ is a commutative Frobenius $\mathbb{K}$-algebra and viceversa. The main result that we are going to prove in this subsection says a little bit more. Suppose that F is a bidimensional TQFT w.r.t $\mathbb{K}$ and suppose that (V, f, g, h, i) is the corresponding commutative Frobenius $\mathbb{K}$-algebra. Intuitively the main result says that the behaviour of F w.r.t. all the other bidimensional TQFTs w.r.t

$\mathbb{K}$ is precisely the behaviour of (V, f, g, h, i) w.r.t. all the other commutative Frobenius $\mathbb{K}$-algebras. In other words, the arrows that link F to the other bidimensional TQFTs w.r.t. $\mathbb{K}$ are in bijective correspondence with the arrows that link (V, f, g, h, i) to the other commutative Frobenius $\mathbb{K}$-algebras.

Definition 4.23. Let $(V, f, g, h, i,)$ and (V', f', g', h', i') be Frobenius $\mathbb{K}$-algebras. A homomorphism of Frobenius $\mathbb{K}$-algebras $(V, f, g, h, i,) \to (V', f', g', h', i')$ is a $\mathbb{K}$-linear map $\varphi\colon V \to V'$ such that $\varphi\colon (V, f, i) \to (V', f', i')$ is a homomorphism of unital associative $\mathbb{K}$-algebras (see Proposition 4.3) and $\varphi\colon (V, g, h) \to (V', g', h')$ is a homomorphism of counital coassociative $\mathbb{K}$-coalgebras (see Definition 4.9).

We indicate as $\textsc{frAlg}(\mathbb{K})$ the category whose objects are the Frobenius $\mathbb{K}$-algebras and whose arrows are the homomorphism of Frobenius $\mathbb{K}$-algebras between them. We indicare as $\textsc{cfrAlg}(\mathbb{K})$ the category whose objects are the commutative Frobenius $\mathbb{K}$-algebras and whose arrows are the homomorphism of Frobenius $\mathbb{K}$-algebras between them, that is, the full subcategory of $\textsc{frAlg}(\mathbb{K})$ spanned by the commutative Frobenius $\mathbb{K}$-algebras.

Remark 4.24 (*Almost the main result*)**.** Let $\mathcal{E}$ be the category whose objects are the symmetric monoidal functors $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma)$ and whose arrows are the monoidal natural transformations between them (see Definition 1.6). In other words $\mathcal{E} := \mathbb{K}\text{-}\textsc{linRep}(\mathcal{S}, \sqcup, \emptyset, \tau)$ (see Definition 1.9). Let $\varphi\colon F \to F'$ be an arrow of $\mathcal{E}$. As usual, let (V, f, g, h, i) and (V', f', g', h', i') be the images of F and F' respectively. Then $(F\emptyset \xrightarrow{\varphi_\emptyset} F'\emptyset) = (\mathbb{K} \xrightarrow{1_\mathbb{K}} \mathbb{K})$. Moreover, for every choice of objects A, B of $\mathcal{S}$, it is the case that $(F(A \sqcup B) \xrightarrow{\varphi_{A\sqcup B}} F'(A \sqcup B)) = (FA \xrightarrow{\varphi_A} GA) \otimes (FB \xrightarrow{\varphi_B} F'B)$. Hence, as every object of $\mathcal{S}$ (different from $\emptyset$) is a finite disjoint union of copies of $\mathbf{1}$, it is the case that every component of φ (different from the $\emptyset$-one) is a finite $\otimes$-paralleling of copies of $\varphi_\mathbf{1}\colon V \to V'$. In particular, it is the case that φ is completely determined by $\varphi_\mathbf{1}$. Finally, as φ is a natural transformation, it is the case that the following diagrams:

$$\begin{array}{cccc}
\begin{tikzcd} F\emptyset \arrow[r, "\varphi_\emptyset"] \arrow[d, "F[lE]"'] & G\emptyset \arrow[d, "G[lE]"] \\ F\mathbf{1} \arrow[r, "\varphi_\mathbf{1}"] & G\mathbf{1} \end{tikzcd} &
\begin{tikzcd} F\mathbf{1} \arrow[r, "\varphi_\mathbf{1}"] \arrow[d, "F[rE]"'] & G\mathbf{1} \arrow[d, "G[rE]"] \\ F\emptyset \arrow[r, "\varphi_\emptyset"] & G\emptyset \end{tikzcd} &
\begin{tikzcd} F\mathbf{1} \arrow[r, "\varphi_\mathbf{1}"] \arrow[d, "F[lF]"'] & G\mathbf{1} \arrow[d, "G[lF]"] \\ F\mathbf{2} \arrow[r, "\varphi_\mathbf{2}"] & G\mathbf{2} \end{tikzcd} &
\begin{tikzcd} F\mathbf{2} \arrow[r, "\varphi_\mathbf{2}"] \arrow[d, "F[rF]"'] & G\mathbf{2} \arrow[d, "G[rF]"] \\ F\mathbf{1} \arrow[r, "\varphi_\mathbf{1}"] & G\mathbf{1} \end{tikzcd}
\end{array}$$

commute. As $\varphi_\mathbf{2} = \varphi_\mathbf{1} \otimes \varphi_\mathbf{1}$ and $\varphi_\emptyset = 1_\mathbb{K}$, this means that $\varphi_\mathbf{1} \circ f = f'$, $g' \circ \varphi_\mathbf{1} = g$, $h' \circ \varphi_\mathbf{1} = (\varphi_\mathbf{1} \otimes \varphi_\mathbf{1}) \circ h$ and $\varphi_\mathbf{1} \circ i = i' \circ (\varphi_\mathbf{1} \otimes \varphi_\mathbf{1})$. In other words, it is the case that $\varphi_\mathbf{1}$ is both a homomorphism of unital associative $\mathbb{K}$-algebras $(V, f, i) \to (V', f', g')$ and a homomorphism of counital coassociative $\mathbb{K}$-coalgebras $(V, g, h) \to (V', g', h')$, that is, it is the case that $\varphi_\mathbf{1}$ is a homomorphism of Frobenius $\mathbb{K}$-algebras $(V, f, g, h, i) \to (V', f', g', h', i')$. Hence there exists a functor $\Psi\colon \mathcal{E} \to \textsc{cfrAlg}(\mathbb{K})$ sending every symmetric monoidal functor $(\mathcal{S}, \sqcup, \emptyset, \tau) \to (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma)$ to its image and every monoidal natural transformation φ between symmetric monoidal functors to its $\mathbf{1}$-component $\varphi_\mathbf{1}$. Corollary 4.22 tells us that this functor is bijective on the objects.

Let F, F' be objects of $\mathcal{E}$ and let $(V, f, g, h, i) := \Psi F$ and $(V', f', g', h', i') := \Psi F'$. Let φ, ψ be arrows $F \to F'$. If $\Psi\varphi = \Psi\psi$, then $\varphi_\mathbf{1} = \psi_\mathbf{1}$, hence $\varphi = \psi$, as, for every $n \in \mathbb{N}$, it is the case that $\varphi_{\boldsymbol{n}}$ and $\psi_{\boldsymbol{n}}$ are $\otimes$-parallelings of n copies of $\varphi_\mathbf{1}$ and $\psi_\mathbf{1}$ respectively. Hence Ψ is faithful.

Moreover, whenever $x\colon V \to V'$ is a homomorphism of Frobenius $\mathbb{K}$-algebras $\Psi F \to \Psi F'$, it is the case that the four squares of before commute with $1_\mathbb{K}$ in place of $\varphi_\emptyset$, x in place of $\varphi_\mathbf{1}$ and $x \otimes x$ in place of $\varphi_\mathbf{2}$. Let $\psi_{\boldsymbol{n}}$ be the $\otimes$-paralleling of n-copies of x for every $n \in \mathbb{N} \setminus \{0\}$

and let $\psi_\emptyset = 1_{\mathbb{K}}$. As the class S (see Theorem 3.4) is a generating class of $(\mathcal{S}, \sqcup, \emptyset, \tau)$ and as the components of ψ commute with the elements of S (this is the commutativity of the squares of before with ψ in place of φ), it is the case that the components of ψ commute with all the arrows of $\mathcal{S}$, that is, ψ is a natural transformation $F \to F'$. Moreover ψ is monoidal by its own definition, hence it is an arrow $F \to F'$ of $\mathcal{E}$. Finally, since $\psi_{\mathbf{1}} = x$ and since x was an arbitrary arrow $\Psi F \to \Psi F'$ of $\textsc{cfrAlg}(\mathbb{K})$, it is the case that Ψ is full.

As F is fully faithful and bijective on the objects, we conclude that it is an isomorphism of categories $\mathcal{E} \cong \textsc{cfrAlg}(\mathbb{K})$.

Theorem 4.25 (*The main result*)**.** *There is an equivalence of categories* $2\textsc{tQft}(\mathbb{K}) \simeq \textsc{cfrAlg}(\mathbb{K})$.

Proof. By Remark 4.24, it suffices to prove that there is an equivalence of categories $2\textsc{tQft}(\mathbb{K}) \simeq \mathcal{E}$. Remind that:

$$2\textsc{tQft}(\mathbb{K}) = \textsc{SymMonCat}((2\textsc{Cob}, \sqcup, \emptyset, \tau), (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma))$$

(see Definition 2.11) and that:

$$\mathcal{E} = \textsc{SymMonCat}((\mathcal{S}, \sqcup, \emptyset, \tau), (\mathbb{K}\text{-}\textsc{Vect}, \otimes, \mathbb{K}, \sigma))$$

(see Remark 4.24 and Definition 1.9). Moreover we remind that $\mathcal{S}$ is a skeleton of $2\textsc{Cob}$, hence in particular the inclusion functor $\mathcal{S} \overset{\iota}{\hookrightarrow} 2\textsc{Cob}$ is an equivalence of categories. Clearly it preserves $\sqcup$, $\emptyset$ and τ, hence it is a symmetric monoidal functor. Therefore there is a functor $\Omega\colon 2\textsc{tQft}(\mathbb{K}) \to \mathcal{E}$ such that $\Omega F = F \circ (\mathcal{S} \overset{\iota}{\hookrightarrow} 2\textsc{Cob})$ for every object F of $2\textsc{tQft}(\mathbb{K})$ and such that every arrow $\{\varphi_A\}_{(A \text{ object of } 2\textsc{Cob})} = \varphi$ of $2\textsc{tQft}(\mathbb{K})$ is sent to its restriction $\{\varphi_A\}_{(A \text{ object of } \mathcal{S})} = \varphi \circ (\mathcal{S} \overset{\iota}{\hookrightarrow} 2\textsc{Cob})$. Let us pick a pseudoinverse $p\colon 2\textsc{Cob} \to \mathcal{S}$ of $\mathcal{S} \overset{\iota}{\hookrightarrow} 2\textsc{Cob}$ (it exists because $\mathcal{S} \to 2\textsc{Cob}$ is an equivalence of categories), that is, there are natural isomorphisms $p \circ \iota \cong 1_{\mathcal{S}}$ and $\iota \circ p \cong 1_{2\textsc{Cob}}$. It is the case that p is naturally isomorphic to a symmetric monoidal functor (see [1]). Hence we can assume that p is a symmetric monoidal functor as well. Moreover we can assume that the natural isomorphisms $a\colon p \circ \iota \cong 1_{\mathcal{S}}$ and $b\colon \iota \circ p \cong 1_{2\textsc{Cob}}$ are monoidal (see [1]). Let $\Lambda\colon \mathcal{E} \to 2\textsc{tQft}(\mathbb{K})$ be such that $\Lambda G = G \circ p$ for every object G of $\mathcal{E}$ and $\Lambda \psi = \psi \circ p$ for every arrow ψ of $\mathcal{E}$.

For every object G of $\mathcal{E}$, it is the case that $a^*_G\colon \Omega\Lambda G = G \circ p \circ i \overset{a}{\cong} G \circ 1_{\mathcal{S}} = G$ is a monoidal natural isomorphism, that is, an isomorphism of $\mathcal{E}$. One can verify that a^*_G is natural in G. Then a^* is a natural isomorphism $\Omega\Lambda \to 1_{\mathcal{E}}$. Analogously, for every object F of $2\textsc{tQft}(\mathbb{K})$, it is the case that $b^*_F\colon \Lambda\Omega F = F \circ i \circ p \overset{b}{\cong} F \circ 1_{2\textsc{Cob}} = F$ is a monoidal natural isomorphism, that is, an isomorphism of $2\textsc{tQft}(\mathbb{K})$. One can verify that b^*_F is natural in F. Then b^* is a natural isomorphism $\Lambda\Omega \to 1_{2\textsc{tQft}(\mathbb{K})}$. Hence we proved that Λ is a pseudoinverse of Ω, that is, Ω is an equivalence of categories. Q.E.D.

A simple generalisation of the main result

Theorem 4.25 is the explicit description of the category of the bidimensional TQFTs that we were looking for. We conclude with the following:

Remark 4.26. In Section 1 we explained how to define a *monoid-object* in a category with finite products. We observed that we are able to do this just because we can talk about monoids by only involving the notions of finite product and composition in $\textsc{Set}$. In fact that means that a category with finite products is precisely what we need in order to talk about a monoid.

Remind that we characterised (Corollary 4.16 and Corollary 4.19) the notion of (commutative) Frobenius $\mathbb{K}$-algebra in such a way that we can talk about it by only involving the (symmetric) monoidal categorical structure of $(\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$. Hence this means that a (symmetric) monoidal categorical structure is everything we need in order to talk about (commutative) Frobenius algebras. Let $(\mathcal{C}, \otimes, \eta, \sigma)$ be a symmetric monoidal category. A *(commutative) Frobenius algebra-object in* $(\mathcal{C}, \otimes, \eta, \sigma)$ is a 5-tuple (V, f, g, h, i), where V is an object of $\mathcal{C}$ and $f\colon \eta(*) \to V$, $g\colon V \to \eta(*)$, $h\colon V \to V \otimes V$ and $i\colon V \otimes V \to V$ are arrows of $\mathcal{C}$, such that the relations $((a)$, $(a'))$, (b), (b'), (c), (c'), (d), (d'), (e) and (e') of the set Φ are verified. Observe that a (commutative) Frobenius $\mathbb{K}$-algebra is just a (commutative) Frobenius algebra-object in $(\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$, precisely as a monoid is nothing but a monoid-object in SET (see Section 1) or as a ring is nothing but a ring-object in SET.

Let $(\mathcal{C}, \otimes, \eta, \sigma)$ be a symmetric monoidal category and let:

$$n\mathrm{TQFT}(\mathcal{C}, \otimes, \eta, \sigma) := \mathrm{SYMMONCAT}((n\mathrm{COB}, \sqcup, \emptyset, \tau), (\mathcal{C}, \otimes, \eta, \sigma))$$

(observe that $n\mathrm{TQFT}(\mathbb{K}) = n\mathrm{TQFT}(\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma))$ for every $n \in \mathbb{N}$.

Observe that the proofs of Corollary 4.22 and Theorem 4.25 do not depend on the fact that $(\mathcal{C}, \otimes, \eta, \sigma) = (\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$. In fact they only depend on the symmetric monoidal structure of $(\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$. We never used in these proofs the non-symmetric monoidal categorical structure of $(\mathbb{K}\text{-}\mathrm{VECT}, \otimes, \mathbb{K}, \sigma)$ (we only used it to prove the characterisations of the notion of (commutative) Frobenius $\mathbb{K}$-algebra provided by Corollary 4.16 and Corollary 4.19, but now this does not matter, since we defined a (commutative) Frobenius algebra-object as a 5-tuple that already satisfies the *right* member of these characterisations). Hence we can repeat the same arguments in order to get that *an object of* $2\mathrm{TQFT}(\mathcal{C}, \otimes, \eta, \sigma)$ *is precisely a commutative Frobenius algebra-object in* $(\mathcal{C}, \otimes, \eta, \sigma)$ and that *there is an equivalence of categories* $2\mathrm{TQFT}(\mathcal{C}, \otimes, \eta, \sigma) \simeq \mathrm{cFRALG}(\mathcal{C}, \otimes, \eta, \sigma)$, being $\mathrm{cFRALG}(\mathcal{C}, \otimes, \eta, \sigma)$ the category whose objects are the commutative Frobenius algebra-objects in $(\mathcal{C}, \otimes, \eta, \sigma)$ and whose arrows are as usual the arrows of $\mathcal{C}$ preserving the structure of commutative Frobenius algebra-object (see Definition 4.23).

Appendix

1. *About the notion of classic model of a given first-order theory.* We briefly present the formal definition of the notion of *classic model of a given first-order theory.* Let us consider a countable set X whose elements will be called *variables.*

Let L be a *first-order language*, that is, a set of function-symbols (each of them with a given arity) and of relation-symbols (again each of them with a given arity). We inductively define the *L-terms* as follows:

(i) every variable x is an L-term;

(ii) whenever n is a natural number, t_1, t_2, ..., t_n are L-terms and f is an n-ary function-symbol of the language L, it is the case that $f(t_1, t_2, ..., t_n)$ is an L-term.

We inductively define the *L-formulas* as follows:

(i) whenever s and t are L-terms, it is the case that $s = t$ is an L-formula;

(ii) whenever n is a natural number, t_1, t_2, ..., t_n are L-terms and R is an n-ary relation-symbol of the language L, it is the case that $R(t_1, t_2, ..., t_n)$ is an L-formula;

(iii) whenever φ and ψ are L-formulas, it is the case that $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ are L-formulas;

(iv) whenever x is a variable and φ is an L-formula, it is the case that $\exists x\varphi$ and $\forall x\varphi$ are L-formulas.

Finally we say that an *L-structure* $\boldsymbol{S}$ is a set S together with: a function $f_S\colon S^n \rightarrow S$ for every n-ary function symbol f of the language L; a relation $R_S \subseteq S^n$ for every n-ary relation symbol R of the language L.

Let $\boldsymbol{S}$ be an L-structure and let v be a map $X \rightarrow S$. Whenever $x \in X$ and $a \in S$ we denote as $v[x/a]$ the unique map $X \rightarrow S$ such that $v[x/a](x) = a$ and $v[x/a](y) = v(y)$ whenever $y \in X \setminus \{x\}$. We inductively define the *$(\boldsymbol{S}, v)$-evaluation* [denoted as $t^{(\boldsymbol{S},v)}$] of an L-term t as follows:

(i) if $t = x$ is a variable, then $t^{(\boldsymbol{S},v)} = x^{(\boldsymbol{S},v)} := v(x)$;

(ii) if n is a natural number, t_1, t_2, ..., t_n are L-terms and f is an n-ary function-symbol of the language L, then $f(t_1, t_2, ..., t_n)^{(\boldsymbol{S},v)} := f_S(t_1^{(\boldsymbol{S},v)}, t_2^{(\boldsymbol{S},v)}, ..., t_n^{(\boldsymbol{S},v)})$.

We say that *the L-structure $\boldsymbol{S}$ together with the map v verifies/models the L-formula* φ [and we denote this as $(\boldsymbol{S}, v) \vDash \varphi$] if:

(i) φ is $s = t$ (for L-terms s and t) and it is the case that $s^{(\boldsymbol{S},v)} = t^{(\boldsymbol{S},v)}$;

(ii) φ is $R(t_1, t_2, ..., t_n)$ [for a natural number n, some L-terms t_1, t_2, ..., t_n and an n-ary relation-symbol R of L] and it is the case that $(t_1^{(\boldsymbol{S},v)}, t_2^{(\boldsymbol{S},v)}, ..., t_n^{(\boldsymbol{S},v)}) \in R_S$;

(iii) φ is $\neg\psi$ [for some L-formula ψ] and it is not the case that $(\boldsymbol{S}, v) \vDash \psi$;

(iv) φ is $\psi_1 \wedge \psi_2$ [for L-formulas ψ_1 and ψ_2] and it is the case that $(\boldsymbol{S}, v) \vDash \psi_1$ and $(\boldsymbol{S}, v) \vDash \psi_2$;

(v) φ is $\psi_1 \vee \psi_2$ [for L-formulas ψ_1 and ψ_2] and it is the case that $(\boldsymbol{S}, \varphi) \vDash \psi_1$ or $(\boldsymbol{S}, \varphi) \vDash \psi_2$;

(*vi*) φ is $\psi_1 \to \psi_2$ [for L-formulas ψ_1 and ψ_2] and it is the case that, whenever $(\boldsymbol{S}, v) \vDash \psi_1$, then $(\boldsymbol{S}, v) \vDash \psi_2$;

(*vii*) φ is $\psi_1 \leftrightarrow \psi_2$ [for L-formulas ψ_1 and ψ_2] and it is the case that $(\boldsymbol{S}, v) \vDash \psi_1$ if and only if $(\boldsymbol{S}, v) \vDash \psi_2$.

(*viii*) φ is $\exists x \psi$ [for some variable x and some L-formula ψ] and there is $a \in S$ such that $(\boldsymbol{S}, v[x/a]) \vDash \psi$;

(*ix*) φ is $\forall x \psi$ [for some variable x and some L-formula ψ] and for every $a \in S$ it is the case that $(\boldsymbol{S}, v[x/a]) \vDash \psi$.

Now, for every L-term t we denote as $FV(t)$ the set of the variables appearing in t (we could define inductively $FV(t)$ for every L-term t) and for every L-formula φ we denote as $FV(\varphi)$ the set of the variables appearing at least once in φ out of the scope of any quantifier appering in φ (again, one could give an inductive definition of $FV(\varphi)$ for every L-formula φ). Then for every L-structure $\boldsymbol{S}$ and for every choice of maps $v, w \colon X \to S$ the following facts hold: *for every L-term t, if $v \restriction_{FV(t)} = w \restriction_{FV(t)}$ then $t^{(\boldsymbol{S},v)} = t^{(\boldsymbol{S},w)}$* and *for every L-formula φ, if $v \restriction_{FV(\varphi)} = w \restriction_{FV(\varphi)}$ then $(\boldsymbol{S}, v) \vDash \varphi$ if and only if $(\boldsymbol{S}, w) \vDash \varphi$.* That means that, whenever an L-formula φ is an *L-sentence*, i.e. $FV(\varphi) = \emptyset$, then: *the fact that $(\boldsymbol{S}, v) \vDash \varphi$ or not does not depend on the choice of the map $v \colon X \to S$.* Hence we say that $\boldsymbol{S}$ *verifies/models* φ [and we denote it as $\boldsymbol{S} \vDash \varphi$] if and only if there is a map $v \colon X \to S$ such that $(\boldsymbol{S}, v) \vDash \varphi$. Finally, an *$L$-theory* is just a set of L-sentences. We say that an L-structure $\boldsymbol{S}$ is a *classic model of a theory T* if and only if $\boldsymbol{S} \vDash \varphi$ for every $\varphi \in T$.

2. *Some clarifications about the categorical notation used during the essay.* Let us recall the notion of *product* between the objects of a category and illustrate the corresponding standard notation (the one that is usually adopted in the literature as well). Let $\mathcal{C}$ be a category and let F be a family of objects of $\mathcal{C}$. We remind that a product of the family F is an object X of $\mathcal{C}$ together with an arrow $p_C \colon X \to C$ of $\mathcal{C}$ for every object C of F, such that the following universal property is satisfied: for every object Y of $\mathcal{C}$ together with an arrow $q_C \colon Y \to C$ of $\mathcal{C}$ for every object C of F, there is unique an arrow $q \colon Y \to X$ such that the following diagram:

$$\begin{tikzcd} Y \arrow[d, "q"'] \arrow[dr, "q_C"] & \\ X \arrow[r, "p_C"'] & C \end{tikzcd}$$

commutes for every object C of F. Applying their universal properties one can easily verify that any two products $(X, \{p_C\}_{(C \text{ in } F)})$ and $(X', \{p'_C\}_{(C \text{ in } F)})$ of F are such that X and X' are isomorphic through a unique isomorphism $f \colon X \to X'$ of $\mathcal{C}$ that commutes with the arrows p_C and p'_C for every object C in F, that is, the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "p_C"] & \\ X' \arrow[r, "p'_C"'] & C \end{tikzcd}$$

commutes for every C in F. Hence, if a product of F exists then it is essentially unique. The category Set has a natural choice of a product for every family of objects of itself: it is just the usual cartesian product between sets with the usual projection maps to each one of its factors. However, this is not always the case. Usually, if a category *has products* (see below) then one assumes that a choice of a product for every family of objects has already been taken. That is why during the essay we always talk about *the* product of a given family of objects of a given category.

Let us assume that the family F of objects of $\mathcal{C}$ is finite. We denote its objects as X_1, X_2 and X_n. During the essay we denote *its* product has $(X_1 \times X_2 \times ... \times X_n, \{\pi_i\}_{i\in\{1,2,...,n\}})$. Moreover, whenever Y is an object of $\mathcal{C}$ and there is an arrow $q_i \colon Y \to X_i$ of $\mathcal{C}$ for every $i \in \{1, 2, ..., n\}$, then during the essay we indicate as $\langle q_1, q_2, ..., q_n\rangle$ the arrow $Y \to X_1 \times X_2 \times ... \times X_n$ whose existence and whose uniqueness are ensured by the universal property of the product $X_1 \times X_2 \times ... \times X_n$. Moreover, if Y_1, Y_2 and Y_n are objects of $\mathcal{C}$ and there is an arrow $f_i \colon Y_i \to X_i$ of $\mathcal{C}$ for every $i \in \{1, 2, ..., n\}$, then during the essay we denoted as $f_1 \times f_2 \times ... \times f_n$ the unique arrow $Y_1 \times Y_2 \times ... \times Y_n$ such that the diagram:

$$\begin{tikzcd}
Y_1 \times Y_2 \times ... \times Y_n \arrow[r, "\pi_i"] \arrow[d, "f_1\times f_2\times ...\times f_n"'] & Y_i \arrow[d, "f_i"] \\
X_1 \times X_2 \times ... \times X_n \arrow[r, "\pi_i"] & X_i
\end{tikzcd}$$

commutes for every $i \in \{1, 2, ..., n\}$ (its existence and uniqueness are ensured by the universal property of $X_1 \times X_2 \times ... \times X_n$ considering the set of arrows $f_i \circ \pi_i \colon Y_1 \times Y_2 \times ... \times Y_n \to X_i$ for every $i \in \{1, 2, ..., n\}$). Observe that we used the same symbol to name the arrows $Y_1 \times ... \times Y_n \to Y_i$ and $X_1 \times ... \times X_n \to X_i$ exhibiting $Y_1 \times ... \times Y_n$ and $X_1 \times ... \times X_n$ as products, as it does not generate any ambiguity. Observe that during the essay we use this notation even if the objects X_i and Y_i are categories and the arrows f_i are functors between them (i.e. even if $\mathcal{C}$ is a category whose objects are categories and whose arrows are functors). If F is empty then *its* product (if it exists) by definition is just an object 1 of $\mathcal{C}$ such that for every object Y of $\mathcal{C}$ there is unique an arrow $Y \to 1$, that in the essay we denote as $!_Y$ (or ! if there is no ambiguity). The object 1 (unique up to unique isomorphism of $\mathcal{C}$) is called *terminal object* of $\mathcal{C}$ if it exists.

We say that *the category* $\mathcal{C}$ *has small products* or simply that it *has products* if and only if, for every family F of objects of $\mathcal{C}$, there is a product of F in $\mathcal{C}$. We say that *the category* $\mathcal{C}$ *has finite products* if, for every finite family F of objects of $\mathcal{C}$, there is a product of F in $\mathcal{C}$. In particular, if $\mathcal{C}$ has finite products then it has *the* terminal object 1. Viceversa, if $\mathcal{C}$ has the terminal object 1 and every family F of object of $\mathcal{C}$ of cardinality 2 has a product, then one can verify that $\mathcal{C}$ has finite products.

Let $\mathbb{K}$ be a field. Sometimes during the essay we invoke the so-called *universal property of the tensor product* of two $\mathbb{K}$-linear spaces. As for every universal property, it characterises (up to some notion of isomorphism) the object that verifies it. Hence we can consider the following definition of tensor product. Let V and W be $\mathbb{K}$-linear spaces. Then a *tensor product* of V and W is a $\mathbb{K}$-linear space T together with a $\mathbb{K}$-bilinear map $f \colon V \times W \to T$ such that the following universal property is satisfied: whenever S is a $\mathbb{K}$-linear space and $g \colon V \times W \to S$ is a $\mathbb{K}$-bilinear map, then there is unique a $\mathbb{K}$-linear map $\overline{g} \colon T \to S$ such that the following diagram:

$$\begin{tikzcd}
V \times W \arrow[r, "g"] \arrow[d, "f"'] & S \\
T \arrow[ur, "\overline{g}"'] &
\end{tikzcd}$$

commutes. As for the product of a family of object in a category (see above), if (T, f) and

(T', f') are tensor products of V and W then there is unique an isomorphism $x: T \to T'$ commuting with f and f' (i.e. such that $x \circ f = f'$).

Let $(T, f: V \times W \to T)$ be a tensor product of V and W. Let $\langle f(V \times W) \rangle$ be the $\mathbb{K}$-linear subspace of T generated by $f(V \times W)$ and let q be the quotient $\mathbb{K}$-linear map $T \to T/\langle f(V \times W) \rangle$. Then $(V \times W \xrightarrow{f} T \xrightarrow{q} T/\langle f(V \times W) \rangle) = (V \times W \xrightarrow{0} T/\langle f(V \times W) \rangle)$, that is, the following diagram:

$$\begin{tikzcd}
V \times W \arrow[r, "0"] \arrow[d, "f"'] & T/\langle f(V \times W) \rangle \\
T \arrow[ur, "q"'] &
\end{tikzcd}$$

commutes. But the following diagram:

$$\begin{tikzcd}
V \times W \arrow[r, "0"] \arrow[d, "f"'] & T/\langle f(V \times W) \rangle \\
T \arrow[ur, "0"'] &
\end{tikzcd}$$

commutes as well. Hence, by the universal property of the couple (T, f), it is the case that $q = 0$ i.e. $\langle f(V \times W) \rangle = T$.

It is the case that a tensor product of two vector spaces V and W always exists. Then, as before (see above), while working with the category $\mathbb{K}$-Vect for some field $\mathbb{K}$, one can assume that a choice of a tensor product for every couple of $\mathbb{K}$-linear spaces has already been taken. If V and W are $\mathbb{K}$-linear spaces, then this choice is usually denoted as $(V \otimes W, V \times W \xrightarrow{\otimes} V \otimes W)$. If $x \in V$ and $y \in W$ then the element $\otimes(x, y)$ of $V \otimes W$ is usually denoted as $x \otimes y$. The simple result we proved right above is telling us that $V \otimes W$ is generated by the elements of the form $x \otimes y$ for $x \in V$ and $y \in W$. That is precisely the reason why we only need to verify a given property on the elements of $V \otimes W$ of this form, if our aim is to verify that this property holds for any element of $V \otimes W$. We use this fact a number of times during the essay.

Other categorical properties of the tensor product that we are interested to and that we use a lot during the essay are:

- Again, the universal property of the tensor product $(V \otimes W, \otimes)$ of two $\mathbb{K}$-linear spaces: for every $\mathbb{K}$-linear space S, it bijectively coverts the $\mathbb{K}$-bilinear maps $V \times W \to S$ into $\mathbb{K}$-linear maps $V \otimes W \to S$. Hence during the essay it allows us to completely discuss some important notions into the category $\mathbb{K}$-Vect (together with its symmetric monoidal structure, but without invoking the obscure notion of $\mathbb{K}$-multilinearity, that does not belong to $\mathbb{K}$-Vect at all!).

- The functoriality of $\otimes$. Indeed there is a functor $\mathbb{K}\text{-Vect} \times \mathbb{K}\text{-Vect} \to \mathbb{K}\text{-Vect}$, again denoted as $\otimes$, sending every couple (V, W) of $\mathbb{K}$-linear spaces into their tensor product $V \otimes W$ and every couple $(f: V \to V', g: W \to W')$ of $\mathbb{K}$-linear maps into the unique $\mathbb{K}$-linear map $V \otimes W \to V' \otimes W'$ (use the universal property of the couple $(V \otimes W, \otimes)$) sending the element $x \otimes y$ to the element $f(x) \otimes g(y)$ for every $x \in V$ and $y \in W$. In other words, the couple (f, g) is sent into the unique $\mathbb{K}$-linear map

$V \otimes W \to V' \otimes W'$, denoted as $f \otimes g$, making the following diagram:

$$\begin{tikzcd}
V \times W \arrow[r, "f\times g"] \arrow[d, "\otimes"'] & V' \times W' \arrow[r, "\otimes"] & V' \otimes W' \\
V \otimes W \arrow[urr, "f\otimes g"'] & &
\end{tikzcd}$$

commute. We use the notation "$f \otimes g$" during the whole essay.

3. *Basic category-theoretic notions appering during the essay.* We remind that a *category* $\mathcal{C}$ is a class, whose elements are called *objects* together with a class $\mathcal{C}(A, B)$, for every couple of objects A and B of $\mathcal{C}$, whose elements are called *morphisms* or *arrows from A to B* and denoted as $A \to B$, such that: the classes $\mathcal{C}(A, B)$ and $\mathcal{C}(C, D)$ are disjoint whenever $A \neq C$ or $B \neq D$; whenever A, B, C are objects of $\mathcal{C}$, $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ then there is an arrow $g \circ f \in \mathcal{C}(A, C)$ and the operation $\circ$ is associative; for every object A of $\mathcal{C}$, there is an arrow $1_A \in \mathcal{C}(A, A)$ (we use this notation during all the essay) such that, for every object B of $\mathcal{C}$ and every arrow $f \in \mathcal{C}(A, B)$ and every arrow $g \in \mathcal{C}(B, A)$ it is the case that $f \circ 1_A = f$ and $1_A \circ g = g$. An arrow $f\colon A \to B$ of $\mathcal{C}$ is called *isomorphism* if there is an arrow $g\colon B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

We talk about functors a number of times during the essay, but actually we *never* verify that a claimed functor actually verifies its definition (that is, that it preserves compositions and identities). Hence we should at least recall the definition: a *functor* F between two categories $\mathcal{C}$ and $\mathcal{D}$ is a functional relation $\mathcal{C} \to \mathcal{D}$ sending every object C of $\mathcal{C}$ to an object FC of $\mathcal{D}$ and every arrow $f\colon C \to C'$ to an arrow $Ff\colon FC \to FC'$ in such a way that the compostions and the identity maps are preserved: $F(g \circ f) = Fg \circ Ff$ for every composable arrows f and g of $\mathcal{C}$ and $F1_C = 1_{FC}$ for every object C of $\mathcal{C}$. Whenever F and G are functors $\mathcal{C} \to \mathcal{D}$, we remind that a *natural transformation* $\alpha\colon F \to G$ is a collection $\{\alpha_C\colon FC \to GC\}_{C\in\mathcal{C}}$ of arrows of $\mathcal{D}$ such that the following diagram:

$$\begin{tikzcd}
FC \arrow[r, "\alpha_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\
FC' \arrow[r, "\alpha_{C'}"] & GC'
\end{tikzcd}$$

commutes for every arrow $f\colon C \to C'$ of $\mathcal{C}$.

Of course functors are composable and every category has an identity functor. If $\mathcal{C}$ and $\mathcal{D}$ are categories and F is a functor $\mathcal{C} \to \mathcal{D}$, we remind that F is an *isomorphism of categories* if it is an isomorphism in the category whose objects are the categories and whose arrows are the functors between them, that is, if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G = 1_{\mathcal{D}}$ and $G \circ F = 1_{\mathcal{C}}$. Moreover, we say that F is an *equivalence of categories* if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ and $1_{\mathcal{D}}$ are naturally isomorphic (during the essay we indicate this as $F \circ G \cong 1_{\mathcal{D}}$) and $G \circ F$ and $1_{\mathcal{C}}$ are naturally isomorphic ($G \circ F \cong 1_{\mathcal{C}}$) (remind that two functors $\mathcal{C} \to \mathcal{D}$ are *naturally isomorphic* if there is a *natural isomorphism* between them, that is, a natural transformations whose components are isomorphisms). If two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* (i.e. there is an isomorphism between them) then during the essay we denote this as $\mathcal{C} \cong \mathcal{D}$. If they are *equivalent* (i.e. there is an equivalence between them) then we denote this as $\mathcal{C} \simeq \mathcal{D}$. Of course any isomorphism of categories is also an equivalence. Remind that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if and only if: F is *essentially surjective*, that is, for every object D of $\mathcal{D}$ there is an object C of

$\mathcal{C}$ such that FC and D are isomorphic in $\mathcal{D}$; F is *fully faithful*, that is, for every choice of objects C and C' of $\mathcal{C}$, the map $\mathcal{C}(C, C') \ni f \mapsto Ff \in \mathcal{D}(FC, FC')$ is bijective.

Finally, we recall that, given a category $\mathcal{C}$, a *subcategory* $\mathcal{D}$ of $\mathcal{C}$ is a category such that: every object of $\mathcal{D}$ is an object of $\mathcal{C}$; every arrow $C \to C'$ of $\mathcal{D}$ is an arrow $C \to C'$ of $\mathcal{C}$; for every object D of $\mathcal{D}$, the identity arrow 1_D of D in $\mathcal{D}$ is also the identity arrow of D in $\mathcal{C}$; whenever f and g are composable arrows of $\mathcal{D}$, then their composition $f \circ g$ in $\mathcal{D}$ is also their composition in $\mathcal{C}$. Clearly there is a functor $\mathcal{D} \to \mathcal{C}$ sending every object and every arrow of $\mathcal{D}$ to themselves. We say that a subcategory $\mathcal{D}$ of $\mathcal{C}$ is a *full subcategory* of $\mathcal{C}$ when for every choice of objects C, C' of $\mathcal{D}$ it is the case that $\mathcal{D}(C, C') = \mathcal{C}(C, C')$, that is, when the inclusion functor is fully faithful. We say that a full subcategory $\mathcal{D}$ of $\mathcal{C}$ is a *skeleton* of $\mathcal{C}$ when it is the case that every two isomorphic objects of $\mathcal{D}$ are equal and every object of $\mathcal{C}$ is isomorphic to an object of $\mathcal{D}$ (i.e. the inclusion functor is an equivalence of categories). As we discuss during the essay, to give a skeleton of a category is equivalent to classifying its objects up to isomorphism (of that category).

4. Sometimes during the essay we talked about functions between sets by using their set-theoretic notation, which is just a consequence of the concrete and explicit definition of function: a function $f \colon X \to Y$ is a subset $f \subseteq X \times Y$ such that for every $x \in X$ there is unique $y \in Y$ such that $(x, y) \in f$. Hence, to write $(x, y) \in f$ corresponds to writing $y = f(x)$. Sometimes it may be easier to think about a function as a subset of a cartesian product verifying this so-called *property of functionality*.

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